Exercises from Multiobjective Optimization

(Milan Hladík, April 17, 2018)

(A) Efficient solutions

1. Consider a multiobjective optimization problem and its efficient solution set $\mathcal{E}$. How can $\mathcal{E}$ change provided we involve an additional objective function $f_{s+1}(x)$? Can it be the same / strict subset / strict superset of $\mathcal{E}$? **4**

2. Find examples for the following situations in multiobjective optimization:

   (a) $\mathcal{E} \not\subseteq \bigcup_{\lambda \in \overline{S}} M_{opt}(\lambda)$, **2**

   (b) $\bigcup_{\lambda \in S} M_{opt}(\lambda) \subsetneqq \mathcal{E}^p$. **2**

3. Find an interesting real life problem or an application from multiobjective optimization, and write its (more-or-less formal) mathematical formulation. **4**

(B) Convex multiobjective optimization

1. We proved
$$\mathcal{E} \subseteq \bigcup_{\lambda \gneqq 0} M_{opt}(\lambda)$$
for convex multiobjective problems. Extend this result for as large as possible class of multiobjective problems. (hint: Where and how did we utilized convexity in the proof?) **4**

2. Consider the following scalarization:
$$\min_{x \in M} \sum_{i=1}^{s} \lambda_i f_i(x)^t,$$
where $\lambda_i > 0$, $\sum_{i=1}^{s} \lambda_i = 1$ and $t > 0$. Under which assumptions each optimal solution of this problem is an efficient solution of the original multiobjective problem
$$\min_{x \in M} \, (f_1(x), \dots, f_s(x))?$$ **4**

3. Prove or disprove: $\mathcal{E}^p = \emptyset \;\Rightarrow \mathcal{E} = \emptyset$. **4**

(C) Linear multiobjective optimization

1. Can it happen that all vertices of a given face F are efficient, but the whole face is not a subset of $\mathcal{E}$? Can this happen for a face of dimension 1 (i.e., for an edge)? **2**

2. Propose a method to check whether a given face is a subset of $\mathcal{E}$. Consider both cases, a bounded and an unbounded face. **6**

3. A feasible point $x^0 \in M$ is *weakly efficient* if there is no $x \in M$ such that $f(x) < f(x^0)$, i.e., $f_i(x) < f_i(x^0)\ \forall i$. Classify the set of weakly efficient solutions $\mathcal{E}^w$ in the hierarchy of inclusions with respect to $M_{opt}(\lambda)$. In particular, consider the general case, the convex case and the linear case of multiobjective programming. **10**

(D) Methods

1. Analyse the following scalarization

$$\min_{x \in M} \max_{i=1,\ldots,s} f_i(x).$$

In particular, find out (or under which assumptions) an optimal solution is efficient for the original problem. Further, inspect properties for the linear and convex cases. **6**

2. Find out whether the the following scalarizations yield efficient solutions (or under which assumptions they do)

   (a) For $y_i := \max_{x \in M} f_i(x)$ and $\lambda > 0$:

$$\max \prod_{k=1}^{s} (y_k - f_k(x))^{\lambda_k} \quad \text{subject to} \quad x \in M,\ f(x) \le y.$$ **4**

   (b) For any $r, d \in \mathbb{R}^s$:

$$\max\ \alpha \quad \text{subject to} \quad x \in M,\ f(x) - \alpha d + f = r,\ f \ge 0.$$ **4**

(E) Numerical exercises

For auxiliary calculations you can use a suitable program.

1. Consider a multiobjective linear program

$$\min\,(-6x_1 - 4x_2, -x_1) \quad \text{subject to} \quad x_1 + x_2 \le 100,\ 2x_1 + x_2 \le 150,\ x_1, x_2 \ge 0.$$

   (a) Find the efficient solution set $\mathcal{E}$ and its image $f(\mathcal{E})$. **2**

   (b) Find the ideal point F and apply the global objective function method using the p-norm with $p = \infty$. **2**

   (c) Solve the problem by ε-constraint method for $r = 2$ and $\varepsilon = 0$. Is the resulting solution efficient? **2**

2. Consider three points $a = (1, 1)$, $b = (1, 4)$ and $c = (4, 4)$ in the plane. Our aim is to find a point minimizing Euclidean distance to all three given points.

   (a) Find the efficient solution set $\mathcal{E}$. **2**

   (b) Solve the problem under the condition that I prefer to be closer to a twice more than closer to the point b, but twice less than to the point c. **2**

   (c) Solve the problem under the condition that preferences on nearness to the points are the same. Could you generalize your results? **2**

(F) Kombinatorická VO

1. Discuss various approaches to solving a knapsack problem with multiple objectives. **4**

2. Propose a method to solve bicriteria TSP (Travelling Salesman Problem), in which the first objective is the classical minimization of the sum of weights of the edges on the route, and the second objective is to minimize the largest edge on the route. **4**