On the Value of Infinite Matrix Games with Interval Payoff

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Optimalizační seminář -Univerzita Karlova
Prague/ CZECHIA

June 7, 2021

Agenda

Introduction to the Domain

The conventional game theory is concerned with how rational individuals make decisions when they are faced with known payoffs. In the real world, however sometimes the payoffs are not known and have to be estimated, and sometimes the payoffs are only approximately known. Hence, outcomes of a matrix game may not be a fixed number. Therefore, interval matrix whose entries are closed intervals by suggested by many researchers to model under uncertainty.

The most comprehensive work dealing with interval analysis was carried out by Moore (1979). Moore (1979) firstly developed an order relation for discrete intervals by generalizing the order relation of real numbers. Then, Ishibuchi and Tanaka (1990) defined two partially ordered relations, Sengupta et al. (1997), Sengupta and Pal (2000) ordered intervals by using acceptability function.

In applications, interval analysis provides rigorous enclosure of solutions to model equations. Moreover, Li (2011) developed a simple and an effective linear programming method for solving matrix games in which the payoffs are expressed with intervals. Li formulated a pair of auxiliary linear programming models to obtain the upper bound and the lower bound of the value of the interval-valued matrix game by using the upper bounds and the lower bounds of the payoff intervals, respectively.

In other respects, infinite games play an impotant role in game theory literature. Marchi (1967) found a necessary and sufficient condition for an infinite matrix game to have a value. And Tijs (1975) introduced various mixed extensions for these games and tackle the problem of the existence of values. Naya (2001) considered infinite matrix whose defining matrix is bounded and such that each row converges to the same real number β and each column to the same real number α . Others results about the existence of a value in infinite two-person zero-sum games obtained by Gomez (1988), Cegielski (1991) and Naya (1996).

Assume that $S_I = \{I_1, I_2, ..., I_m\}$ and $S_{II} = \{II_1, II_2, ..., II_n\}$ are sets of the pure strategies for the players I and II respectively. If the player I adopts any pure strategy $I_i \in S_I$ and the player II adopts any pure strategy $II_j \in S_{II}$, then the payoff of the player I is expressed with an interval $\widetilde{g}_{ij} = [g_{ijL}, g_{ijR}]$ and G concisely is expressed in the matrix form as follows:

$$G = \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_m \end{array} \begin{array}{c} \begin{array}{cccc} II_1 & II_2 & \cdots & II_n \end{array} \\ \begin{bmatrix} [g_{11L}, g_{11R}] & [g_{12L}, g_{12R}] & \cdots & [g_{1nL}, g_{1nR}] \\ [g_{21L}, g_{21R}] & [g_{22L}, g_{22R}] & \cdots & [g_{2nL}, g_{2nR}] \\ \vdots & \vdots & \cdots & \vdots \\ [g_{m1L}, g_{m1R}] & [g_{m1L}, g_{m1R}] & \cdots & [g_{mnL}, g_{mnR}] \end{bmatrix} \end{array}$$

Li, D.F., 2016. Linear Programming Models and Methods of Matrix Games with Payoffs of Triangular Fuzzy Numbers, Springer-Verlag Berlin Heidelberg.

Interval matrix games and their solutions

- Sets of mixed strategies for Players I and II are denoted by X and Y, where

$$X = \left\{ x = (x_1, x_2, ..., x_m) \,\middle|\, \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0\ (i = 1, 2, ..., m) \right\}$$

and

$$Y = \left\{ y = (y_1, y_2, ..., y_n) \,\middle|\, \sum_{j=1}^{n} y_i = 1,\ y_j \geq 0\ (j = 1, 2, ..., n) \right\}$$

respectively.

Li, D.F., et al. 2012. Interval programming models for matrix games with interval payoffs, Optimization Methods and Software, 27(1), 1-16.

Without loss of generality, the payoff matrix for Player I is concisely expressed in the interval form as follows:

$$G = \left(\widetilde{g}_{ij}\right)_{mxn}.$$

Then, a matrix game with interval payoffs is expressed with the triplet as follows:

$$IG = \{X,\ Y,\ G\}.$$

Interval matrix games and their solutions

- The expected payoff of Player I can be computed as follows:

$$h(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{a}_{ij}\ x_i\ y_j = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ijL}\ x_i\ y_j\ ,\ \ \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ijR}\ x_i\ y_j\right]\ ,$$

  which is an interval.

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Interval matrix with saddle point

The position (k, r) of the interval payoff matrix G was defined as a saddle point of the interval matrix game G if

$$V_L = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} \{[a_{ijL}, a_{ijR}]\}$$
$$V_U = \min_{1 \leq j \leq n} \max_{1 \leq i \leq m} \{[a_{ijL}, a_{ijR}]\}$$

exist and equal, i.e.,

$$[a_{krL}, a_{krR}] = V_L = V_U$$

Li, D.F., 2011. On Notes "Linear Programming Tecnique To Solve Two-Person Matrix Games With Interval Payoffs, Asia Pacific J of Operation Research, 28(6), 705-737.

Example

Let's consider the $2x2$ interval matrix game with the interval payoff matrix as follows

$$G_1 = \begin{array}{c} \\ I_1 \\ I_2 \end{array} \begin{array}{c} \begin{array}{cc} II_1 & II_2 \end{array} \\ \begin{pmatrix} [-1,1] & [1,3] \\ [0,2] & [-2,0] \end{pmatrix} \end{array}$$

Obviously,

$$\max_{1 \le i \le 2} \min_{1 \le j \le 2} \{[a_{ijL}, a_{ijR}]\} = [a_{11L}, a_{11R}] = [-1, 1]$$

Nayak, P.K. and Pal, T.K., 2009. Linear Programming Tecnique To Solve Two-Person Matrix Games With Interval Payoffs, Asia Pacific J of Operation Research, 26(2), 285-305.

Example (Continue)

$$\min_{1 \leq j \leq 2} \max_{1 \leq i \leq 2} \{[a_{ijL}, a_{ijR}]\} = [a_{21L}, a_{21R}] = [0, 2] .$$

Hence,

$$\max_{1 \leq i \leq 2} \min_{1 \leq j \leq 2} \{[a_{ijL}, a_{ijR}]\} \neq \min_{1 \leq j \leq 2} \max_{1 \leq i \leq 2} \{[a_{ijL}, a_{ijR}]\} .$$

Therefore, the interval matrix game G_1 has no saddle point in pure strategies.

Interval matrix games and their solutions

Assume that Player I is a maximizing player and Player II is a minimizing player. That is to say, Player II is interested in finding a mixed strategy $y \in Y$ so as to minimize $h(x, y)$, denoted by

$$\min_{y \in Y} h(x, y).$$

Hence, Player I should choose a mixed strategy $x \in X$ that maximizes $\min_{y \in Y} h(x, y)$ of Player II, that is,

$$\bar{v}^* = \max_{x \in X} \min_{y \in Y} h(x, y)$$

which is called Player I's gain-floor.

Similarly, Player I is interested in finding a mixed strategy $x \in X$ so as to minimize $h(x, y)$, denoted by

$$\max_{x \in X} h(x, y).$$

Thus, Player II should choose a mixed strategy $y \in Y$ that minimizes $\max_{x \in X} h(x, y)$ of Player I, that is,

$$\bar{\omega}^* = \min_{y \in Y} \max_{x \in X} h(x, y)$$

which is called Player II's loss-ceiling.

Li, D.F., et al. 2012. Interval programming models for matrix games with interval payoffs, Optimization Methods and Software, 27(1), 1-16.

Definition

Let

$$\bar{v} = [v_L,\ v_R] \quad \text{and} \quad \bar{\omega} = [\omega_L,\ \omega_R]$$

be two intervals on the real number set $\mathbb{R}$. Assume that there exist strategies $x^* \in X$ and $y^* \in Y$. If for any strategies $x \in X$ and $y \in Y$, $(x^*, y^*, \bar{v}, \bar{\omega})$ satisfies both

$$x^* G\ y^T \geq\ \bar{v} \quad \text{and} \quad x\ G\ {y^*}^T \leq\ \bar{\omega}$$

then $(x^*, y^*, \bar{v}, \bar{\omega})$ is called reasonable solution of the interval matrix game G.

Li, D.F., 2011. On Notes "Linear Programming Tecnique To Solve Two-Person Matrix Games With Interval Payoffs, Asia Pacific J of Operation Research, 28(6), 705-737.

- $\bar{v}$ and $\bar{\omega}$ are called reasonable values for Players I and II, respectively; x^* and y^* are called reasonable strategies for Players I and II, respectively. However, the reasonable solution is not the solution of the interval matrix game. So the concept of the solution of the interval matrix game is given in the following definition.
- Let U and W be the sets of reasonable values $\bar{v}$ and $\bar{\omega}$ for Players I and II, respectively.

Definition

Assume that there exist two reasonable values $\bar{v}^* \in U$ and $\bar{\omega}^* \in W$. If there do not exist reasonable values $\bar{v}^{'} \in U$ and $\bar{\omega}^{'} \in W$ such that they satify both

$$\bar{v}^{'} \geq \bar{v}^* \text{ and } \bar{\omega}^{'} \leq \bar{\omega}^*$$

then $\left(x^*, y^*, \bar{v}^*, \bar{\omega}^*\right)$ is called a solution of the interval matrix game G; x^* is called an optimal stretagy for Player I and y^* is called an optimal strategy for Player II ; $\bar{v}^*$and $\bar{\omega}^*$are called Player I 's gain-floor and Player II 's loss-ceiling, respectively.

Li, D.F., 2011. On Notes "Linear Programming Tecnique To Solve Two-Person Matrix Games With Interval Payoffs". Asia Pacific J of Operation Research. 28(6). 705-737.

Theorem

Both $\bar{v}^* = \max_{x \in X} \min_{y \in Y} \left\{ xGy^T \right\}$ *and* $\bar{\omega}^* = \min_{y \in Y} \max_{x \in X} \left\{ xGy^T \right\}$ *exist and the following inequality is valid*

$$\bar{v}^* \leq \bar{\omega}^*.$$

Li, D.F., 2011. On Notes "Linear Programming Tecnique To Solve Two-Person Matrix Games With Interval Payoffs". Asia Pacific J of Operation Research. 28(6). 705-737.

Thus far we have dealt with finite games, i.e., games in which each player has a finite number of strategies. Let's us consider now a generalization to infinite games. Such a games the players have countably many pure strategies. As in the case of finite games, if Player I chooses his i th pure strategy, while Player II chooses his j th pure strategy, then Player I receives payoff a_{ij} in which a_{ij} is the entry of i th row and j th column of matrix.

- A mixed strategy for I, in this case, will be a sequence $(x_1, x_2, \ldots)$ satisfying

$$\sum_{i=1}^{\infty} x_i = 1, \; x_i \geq 0.$$

Owen, G., 1995. Game Theory Third Edition Academic Press.

- A mixed strategy for II will be a sequence $(y_1, y_2, \dots)$ defined similarly, The payoff function for the mixed strategies (x, y) will be defined as follows

$$h(x, y) = \sum_{i=1}^{\infty} x_i \; a_{ij} \; y_j .$$

Owen, G., 1995. Game Theory Third Edition Academic Press.

Infinite interval matrix game

Infinite interval matrix game is the interval generation of classical infinite matrix games. Let $G = (\widetilde{g}_{ij})$ be an infinite interval matrix with upper bound

$$K = \sup\{|\widetilde{g}_{ij}| : i, j \in \mathbb{N}\}$$

Let

$$S = \{x \in \mathbb{R}^N : \sum_{i=1}^{\infty} x_i = 1, x_i \geq 0, \forall i \in \mathbb{N}\}$$

is the set of strategies of both players.

Owen, G., 1995. Game Theory Third Edition Academic Press.

Definition

If the first player chooses mixed strategy x while the second player chooses mixed strategy y, then the expected payoff can be written as

$$h(x,y) = xGy^T, \ \forall (x,y) \in S \times S$$

$$\widetilde{V}_L = \sup_{x\in X} \inf_{y\in Y} h(x,y) \quad \widetilde{V}_U = \inf_{y\in Y} \sup_{x\in X} h(x,y)$$

to be lower and upper value of infinite interval matrix game, respectively.

Naya, L.M., 2001. On The Value Of Some Infinite Matrix Games. Math. of Operations Research,.26(1). 82-88.

Interval Vectors

An n-dimensional interval vector is an ordered n-tuple of intervals,

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_m) = ([x_{1L}, x_{1R}], [x_{2L}, x_{2R}], ..., [x_{nL}, x_{nR}])$$

Let $\widetilde{x} = [x_L, x_R] \in \widetilde{\mathbb{R}}^n$, where $\widetilde{\mathbb{R}}^n$ be the set of n-dimensional interval vector, then the absolute value of interval is defined as follows:

$$|\widetilde{x}| = \max\{|x_L|, |x_R|\}.$$

If $|\widetilde{x}| = 0$, then $\widetilde{x}$ is said to be a zero interval. If the absolute value of each element of $\widetilde{x}$ is zero, then $\widetilde{x}$ is a zero interval vector is denoted by $\widetilde{0}$.

Chiao, K.P., 2002. Fundamental Properties of Interval Vector Max-Norm, Tamsui Oxf J Math Sci 18(2):219-233.

Definition

The sequence of intervals $(\widetilde{x}_n)$ is convergent to a bounded interval $\widetilde{x} = [x_L, x_R]$ if

$$\lim_{n\to\infty} x_{n_L} = x_L \text{ and } \lim_{n\to\infty} x_{n_R} = x_R \tag{1}$$

(1) is equivalent to

$$\lim_{n\to\infty} \widetilde{x}_n = \widetilde{x}$$

Chiao, K.P., 2002. Fundamental Properties of Interval Vector Max-Norm, Tamsui Oxf J Math Sci 18(2):219-233.

Suppose $(\widetilde{x}_n)$ is convergent sequence of real numbers with limit x, then it is easy to verify that

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} |x_n - x| = 0 \tag{2}$$

(2) can be extended for the sequence of intervals.

Theorem

Let $(\widetilde{x}_n)$ be a convergent sequence of intervals with limit interval $\widetilde{x}$ then

$$\lim_{n\to\infty} \widetilde{x}_n = \widetilde{x} \Leftrightarrow \lim_{n\to\infty} |\widetilde{x}_n - \widetilde{x}| = 0.$$

Chiao, K.P., 2002. Fundamental Properties of Interval Vector Max-Norm, Tamsui Oxf J Math Sci 18(2):219-233.

Concluding Remarks

Let $G = (\widetilde{g}_{ij})_{i,j \in \mathbb{N}}$ be a bounded infinite interval matrix, so

$$T_i = \inf_{m \geq 1} \sup_{j \geq m} \widetilde{g}_{ij}, \ \forall j \in \mathbb{N} \ ve \ T = (T_i)_{i \in \mathbb{N}}$$

$$K_j = \sup_{m \geq 1} \inf_{i \geq m} \widetilde{g}_{ij}, \ \forall i \in \mathbb{N} \ \ ve \ K = (K_j)_{j \in \mathbb{N}}$$

Now, we can give the following outcomes which characterize the value of the infinite interval matrix game.

(1) Let $G = (\widetilde{g}_{ij})$ be a bounded infinite interval matrix. If $G = (\widetilde{g}_{ij})$ has a value then

$$\inf_{j \in \mathbb{N}} K_j \leq \sup_{i \in \mathbb{N}} T_i$$

(2) If $G = (\widetilde{g}_{ij})$ is a bounded infinite interval matrix such that its sequence of rows converge to an interval number $\widetilde{\alpha}$ and its sequence of columns converge to the interval number $\widetilde{\beta}$, then if $\widetilde{\alpha} = \widetilde{\beta}$, $G = (\widetilde{g}_{ij})$ has a value and $\widetilde{\alpha} = \widetilde{\beta} = \widetilde{v}$.

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