# Topics in Sparse Recovery via Constrained Optimization: Least Sparsity, Solution Uniqueness, and Constrained Exact Recovery

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# Outline

**Introduction to Compressed Sensing**

(T1) **Least Sparse Solution of $p$-norm Optimization with $p > 1$**

(T2) **Solution Uniqueness to Problems Involving Convex PA Functions with Applications to Constrained $\ell_1$-Minimization**

- Solution Uniqueness Conditions to Basis Pursuit-like, LASSO-like and Other Important Problems
- Verification of Solution Uniqueness Conditions

(T3) **Exact Recovery over Constraint Sets using Matching Pursuit Algorithm**

- Constrained Matching Pursuit
- Coordinate-Projection Admissible Sets
- Exact Recovery Conditions on Coordinate-Projection Admissible Sets

# Sparsity Model and Original Compressed Sensing Problem

| | |
|---|---|
| **Sparsity:** | Most of components are zero |
| **Sparsity Level:** | Number of nonzero entries |
| **Compressiblity:** | Well-approximated by sparse signals |

## Compressed Sensing

To recover a sparse vector $x \in \mathbb{R}^N$ from a measurement vector $y \in \mathbb{R}^m$ with $y = Ax$ (possibly subject to errors) and $A \in \mathbb{R}^{m \times N}$ $(m \ll N)$ is a measurement matrix.

## Problem Formulation

Let $\|x\|_0 :=$ Card$(x)$, the original CS problem can be modeled below:

$$\min_{x \in \mathbb{R}^N} \quad \|x\|_0 \qquad \text{subject to} \qquad y = Ax \qquad (P_0).$$

## Applications

Engineering, Statistics, Signal and Image Processing, and etc.

# Algorithms in Compressed Sensing

**Sparsity based Optimization Algorithms**

Since $\ell_p \to \ell_0$ as $p \downarrow 0$, one can approximate $(P_0)$ by the following:

$$\min_{x \in \mathbb{R}^N} \quad \|x\|_p \qquad \text{subject to} \qquad y = Ax.$$

**Greedy Algorithms**

They directly tackle the original problem by making a local optimal decision at each step with an attempt to find a global optimal solution.

**Thresholding based Algorithms**

Most of them solve the square system $A^T Ax = A^T y$ through a fixed-point method and exploit hard thresholding operator.

# (Topic 1) Main Problems Used in Sparse Optimization

## Generalized Basis-Pursuit

$$\min_{x \in \mathbb{R}^N} \|x\|_p \quad \text{subject to} \quad Ax = y$$

## Generalized Basis-Pursuit Denoising I

$$\min_{x \in \mathbb{R}^N} \|x\|_p \quad \text{subject to} \quad \|Ax - y\|_2 \leq \epsilon$$

## Generalized Basis-Pursuit Denoising II

$$\min_{x \in \mathbb{R}^N} \|Ax - y\|_2 \quad \text{subject to} \quad \|x\|_p \leq \eta$$

## Generalized Ridge Regression

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2 + \lambda \|x\|_p^p$$

## Generalized Elastic Net

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2 + \lambda_1 \|x\|_p^r + \lambda_2 \|x\|_2^2$$

# Geometry of $\text{BP}_p$ and $\text{BPDN}_p$ for Different $p$'s

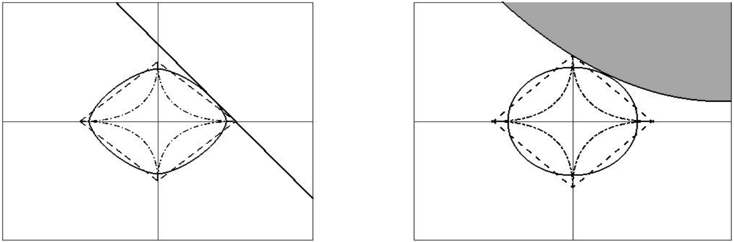


Figura: Geometries of BP and BPDN for different values of $p$

## A Geometrical Illustration

$0 < p < 1 \longrightarrow$ A good choice but nonconvex!

$p = 1 \longrightarrow$ A good choice and results in a convex program!

$p > 1 \longrightarrow$ Not a good choice! How bad?

# Main Results on Least Sparsity with $p > 1$

**Proposition**

*Let $p > 1, 0 < \eta < \min_{Ax=y} \|x\|_p, \lambda > 0,\ \ r \geq 1, \lambda_1 > 0$ and $\lambda_2 > 0$. Each of the above optimization problems attains a unique optimal solution for any $A$ and $y$ as long as the associated constraint sets are nonempty.*

Consider this open set in $\mathbb{R}^{m\times N} \times \mathbb{R}^m$ whose complement has measure zero: $S := \{(A, y) \mid \text{each } m \times m \text{ submatrix of A is invertible and } y \neq 0\}$.

**Theorem**

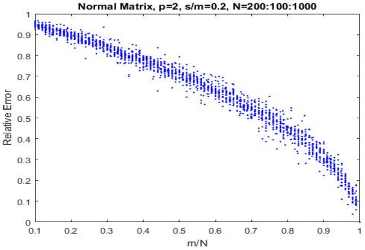
*Let $p > 1, N \geq 2m - 1, 0 < \epsilon < \|y\|_2, 0 < \eta < \min_{Ax=y} \|x\|_p, \lambda > 0$, $r \geq 1,\ \lambda_1 > 0$ and $\lambda_2 > 0$. For almost all $(A, y) \in \mathbb{R}^{m\times N}$, the unique optimal solution to the any of the above problems has a support size of $N$.*

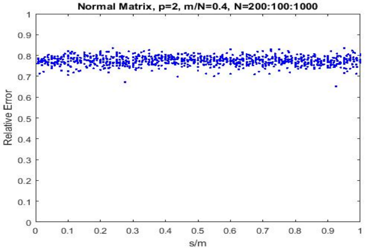
# Methodologies and Compressibility

## Summary of Methodologies

- Using KKT conditions and implicit function theorem, we prove that $x^*$, possibly along with a Lagrange multiplier is a $C^1$ function of $(A, y)$ on the set $S$.
- For each $i = 1, \ldots, N$, if $x_i^*$ is vanishing at $(A, y) \in S$, then its gradient evaluated at $(A, y)$ is nonzero.
- The zero set of each component $x_i^*$ has zero measure.

## Compressiblity



# (Topic 2) $\ell_1$-Norm based Optimization

In sparse recovery, the desired vector is often a solution for one of the following problems:

$$\min_{x\in\mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad Ax = y \tag{BP}$$

$$\min_{x\in\mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad \|Ax - y\|_2 \leq \epsilon \tag{BPD I}$$

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2 \quad \text{subject to} \quad \|x\|_1 \leq \eta \tag{BPD II}$$

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2 + \lambda\|x\|_1 \tag{LASSO}$$

$Note\ that\ \|.\|_1\ is\ not\ strictly\ convex \implies nonunique\ solution!$

**Is this important?**

$$If\ not,\ recovery\ process\ is\ not\ successful!$$

# A Review on Solution Uniqueness (Individual Recovery)

- Foucart established some results for the problems (BP) and (BPD I).
- Zhang et al. established necessary and sufficient conditions for the mentioned problems when $\|.\|_2^2$ is replaced with a strictly convex smooth function. Later, they replaced $\|x\|_1$ by $\|Ex\|_1$.
- Gilbert replaced $\|.\|_1$ with a polyhedral gauge function: A convex piecewise affine function that is nonnegative, positively homogeneous of degree 1, and vanishes at 0.
- Zhao established necessary and sufficient conditions for nonnegative sparse vectors that satisfy an equality linear system.

**Is there a room to improve?**

$Yes!$

# Motivations and Contributions

## Motivations

- To add general linear inequality constraints $\longrightarrow$ Dantzig selector:

$$\min_{x\in\mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad \|A^T(Ax-y)\|_\infty \leq \epsilon.$$

- To go beyond $\|x\|_1$ and $\|Ex\|_1 \rightarrow$ fused LASSO:

$$\min_{x\in\mathbb{R}^N} \|Ax-y\|_2^2 + \lambda_1 \cdot \|x\|_1 + \lambda_2 \cdot \|D_1 x\|_1.$$

- Explicit dual-based conditions $\rightarrow$ easy and computationally favorable.

## Contributions

- Added general linear inequality constraints.
- Considered convex piecewise affine functions, including $\ell_1$-norm.
- Developed a unifying approach that recovers all the known results and enables us to tackle new problems.

# General Framework

Let $A \in \mathbb{R}^{m\times N}, C \in \mathbb{R}^{p\times N}$ and $f : \mathbb{R}^m \to \mathbb{R}$ be a $C^1$ strictly convex function. Further, assume $g(x)$ is a convex piecewise affine function.

**Main Question**

Given a feasible point $x^*$ for any of the below problems, under which conditions this vector is the **unique solution**?

$$\min_{x\in\mathbb{R}^N} g(x) \quad \text{subject to} \quad Ax = y \ \text{ and } \ Cx \geq d \tag{BP-like}$$

$$\min_{x\in\mathbb{R}^N} g(x) \quad \text{subject to} \quad f(Ax-y) \leq \epsilon \ \text{ and } \ Cx \geq d \tag{BPD I-like}$$

$$\min_{x\in\mathbb{R}^N} f(Ax-y) \quad \text{subject to} \quad g_1(x) \leq \eta_1, \dots, \ g_r(x) \leq \eta_r \ \text{ and } \ Cx \geq d \tag{BPD II-like}$$

$$\min_{x\in\mathbb{R}^N} f(Ax-y) + g(x) \quad \text{subject to} \quad Cx \geq d \tag{LASSO-like}$$

# Preliminaries (Convex Piecewise Affine Functions)

Let $g : \mathbb{R}^N \to \mathbb{R}$ be a convex piecewise affine (PA) function:

$$g(x) = \max_{i=1,2,\ldots,l} \left( p_i^T x + \gamma_i \right).$$

For $x^* \in \mathbb{R}^N$ with $Cx^* \geq d$, define $\alpha := \{i \in \{1, \ldots, p\} \mid (Cx^* - d)_i = 0\}$,

$$\mathcal{I} := \left\{ i \in \{1, \ldots, l\} \mid p_i^T x^* + \gamma_i = g(x^*) \right\} \text{ and } W := \begin{bmatrix} p_{i_1}^T \\ \vdots \\ p_{i_{|\mathcal{I}|}}^T \end{bmatrix} \in \mathbb{R}^{|\mathcal{I}| \times N}.$$

Finding the matrix $W$ is equivalent to finding the convex hull generators of $\partial g(x^*)$.

# A Key Lemma

## Lemma

*Let $A \in \mathbb{R}^{m\times N}$ and $H \in \mathbb{R}^{r\times N}$. Then,*

$$\{u \in \mathbb{R}^N \mid Au = 0,\ Hu \geq 0\} = \{0\}$$

*if and only if the following conditions hold:*

*(i)* $\{u \in \mathbb{R}^N \mid Au = 0,\ Hu = 0\} = \{0\}$; *and*

*(ii) There exist* $z \in \mathbb{R}^m$ *and* $z' \in \mathbb{R}^r_{++}$ *such that* $A^T z = H^T z'$.

**Main Idea of Proof**

Define the linear program:

$$\max\ 1^T Hu \quad \text{subject to} \quad Au = 0,\ Hu \geq 0. \qquad (LP)$$

Then, $\{u \in \mathbb{R}^N \mid Au = 0,\ Hu \geq 0\} = \{0\}$ if and only if

(i) $\{u \in \mathbb{R}^N \mid Au = 0,\ Hu = 0\} = \{0\}$; and

(ii') zero is the optimal value of (LP).

# Basis Pursuit-like Problem

**Theorem**

*Let $x^*$ be a feasible point of the optimization problem (BP-like). Then $x^*$ is its unique minimizer if and only if the following conditions hold:*

*(i)* $\{v \in \mathbb{R}^N \mid Av = 0,\ C_{\alpha\bullet} v = 0,\ Wv = 0\} = \{0\}$*; and*

*(ii) There exist* $w \in \mathbb{R}^m, w' \in \mathbb{R}^{|\alpha|}_{++}$*, and* $w'' \in \mathbb{R}^{|\mathcal{I}|}$ *with* $0 < w'' < 1$ *and* $1^T w'' = 1$ *such that* $A^T w - C^T_{\alpha\bullet} w' + W^T w'' = 0$

## Main Steps of Proof

1. For sufficiently small $\|v\|$, we have $g(x^* + v) = g(x^*) + \max_{i \in \mathcal{I}} p_i^T v$.
2. $x^*$ is the unique solution if and only if $v^* = 0$ for

$$\min_{v \in \mathbb{R}^N} \left( \max_{i \in \mathcal{I}} p_i^T v \right) \quad \text{subject to} \quad Av = 0, \quad C_{\alpha\bullet} v \geq 0.$$

3. $v^* = 0$ is the unique solution of this problem if and only if

$$\{v \in \mathbb{R}^N \mid Av = 0,\ C_{\alpha\bullet} v \geq 0,\ \max_{i \in \mathcal{I}} p_i^T v \leq 0\ [\text{or } Wv \leq 0]\} = \{0\}.$$

# How to Verify These Conditions?

Solution uniqueness criteria that we found consist of:

(a) full column rank condition for a matrix $\longrightarrow$ $Linear\ Algebra$

(b) consistency of a linear system with non-strict inequalities $\longrightarrow$ $LP$

(c) consistency of another linear system with strict inequality $\longrightarrow$ ?

## Lemma

*Let $A \in \mathbb{R}^{m\times N}$, $y \in \mathbb{R}^m$ be given. Then, the linear inequality system*

$$Ax = y, \quad x > 0;$$

*has a solution if and only if the following linear program is solvable and attains a positive optimal value:*

$$\max \ \epsilon \qquad \textit{subject to} \qquad Ax = y, \quad x \geq \epsilon.1, \quad \epsilon \leq 1.$$

# (Topic 3) Orthogonal Matching Pursuit (OMP)

The orthogonal matching pursuit is a greedy algorithm to tackle the following problem on $\mathbb{R}^N$:

$$\min_{x \in \mathbb{R}^N} \quad \|x\|_0 \qquad \text{subject to} \qquad Ax = y.$$

**Idea**

Starting with $x_0 = 0$ (set the initial support set $\mathcal{S}_0 = \emptyset$), in the $n$th step, let:

$$j^* \in \text{Argmin}_j \quad \min_t \; \|y - A(x_n + t\mathbf{e}_j)\|_2^2.$$

Add this $j^*$ to the current support index approximation set $\mathcal{S}_n$ and then update the iteration by defining:

$$x_{n+1} \in \text{Argmin}_z \; \|y - Az\|_2^2 \quad \text{subject to} \quad \text{supp}(z) \subseteq \mathcal{S}_{n+1}.$$

# Orthogonal Matching Pursuit

**Orthogonal Matching Pursuit**

| | |
|---|---|
| Input: | measurement matrix $A$, and measurement vector $y$ |
| Initialization: | $x_0 = 0$ and $\mathcal{S}_0 = \emptyset$ |
| Iteration: | repeat until a stopping criteria is met at $n = k$: |
| | $\mathcal{S}_{n+1} = \mathcal{S}_n \cup \{j_{n+1}\}$ where $j^*_{n+1} \in \text{Argmax}_j \lvert (A^T(y - Ax_n))_j \rvert$ |
| | $x_{n+1} \in \text{Argmin}_z \lVert y - Az \rVert_2^2 \quad s.t. \quad \text{supp}(z) \subseteq \mathcal{S}_{n+1}$. |
| Output: | the $k$-sparse vector $x^* = x_k$. |

**Advantage**

Fast at least for relatively small sparsity levels if $A$ is well-chosen.

**Theorem**

*The OMP recovers any $k$-sparse signal $x$ from the measurement $y = Ax$ in at most $k$ iterations if $\delta_k + \sqrt{k}\theta_{k,1} < 1$ (Mo and Shen [2012]).*

## Definition

Given a matrix $A \in \mathbb{R}^{m\times N}$ and a constraint set $\mathcal{P}$, we say that **the exact support recovery** of a vector $x^* \in \Sigma_k \cap \mathcal{P}$ is achieved from $y = Ax^*$ via the OMP (or the CMP), if along an *arbitrary* sequence $\big((x_n, j^*_n, \mathcal{S}_n)\big)_{n\in\mathbb{N}}$ for the given $x^*$, there exists an index $s \in \mathbb{N}$ such that $\mathcal{S}_s = \mathrm{supp}(x^*)$. If the exact support recovery of any vector of $\Sigma_k \cap \mathcal{P}$ is achieved, then we call the exact support recovery on $\Sigma_k \cap \mathcal{P}$ is achieved.

## Definition

Given a matrix $A \in \mathbb{R}^{m\times N}$ and a constraint set $\mathcal{P}$, we say that **the exact vector recovery** of $x^*$ is achieved from $y = Ax^*$ via the OMP (or the CMP) if (i) the exact support recovery of $x^*$ is achieved, and (ii) along *any* sequence $\big((x_n, j^*_n, \mathcal{S}_n)\big)_{n\in\mathbb{N}}$ for the given $x^*$, once $\mathcal{S}_s = \mathrm{supp}(x^*)$ is reached, then the projection step has a *unique* solution $x_s = x^*$.

# Motivations and Contributions

## Motivations

- Using greedy algorithm for constrained recovery.
- Identifying conditions for exact support and vector recovery.
- Developing verifiable sufficient conditions for uniform recovery.

## Contributions

- Introduced a greedy algorithm for constrained problems.
- Defined rigorous notions of exact and vector recovery.
- Analyzed necessary and sufficient conditions for constrained recovery and showed that they critically depend on constraint sets, so we introduced CP admissible sets.
- Used properties of CP admissible to develop verifiable recovery conditions based on RI and RO-like constants.

# Constrained Matching Pursuit

Our goal is to extend the OMP algorithm to solve the following problem:

$$\min_{x\in\mathbb{R}^N} \quad \|x\|_0 \quad \text{subject to} \quad Ax = y \quad \text{and} \quad x \in \mathcal{P}.$$

**Constrained Matching Pursuit (CMP)**

| | |
|---|---|
| Input: | measurement matrix $A$, and measurement vector $y$, and constraint set $\mathcal{P}$ containing the zero vector. |
| Initialization: | $x_0 = 0$ and $S_0 = \emptyset$. |
| Iteration: | repeat until a stopping criteria is met at $n = k$:<br>$\mathcal{S}_{n+1} = \mathcal{S}_n \cup \{j_{n+1}\}$ with $j^*_{n+1} \in \text{Argmin}_{j\in\mathcal{S}_n^c}\ f^*_j$;<br>where $f^*_j = \min_t \|y - A(x_n + t\mathbf{e}_j)\|_2^2 \quad s.t. \quad x_n + t\mathbf{e}_j \in \mathcal{P}$.<br>$x_{n+1} \in \text{Argmin}_{z\in\mathcal{P}} \|y - Az\|_2^2 \quad s.t. \quad \text{supp}(z) \subseteq \mathcal{S}_{n+1}$. |
| Output: | the $k$-sparse vector $x^* := x_k$. |

# Recovery Conditions for CMP

For any $u, v \in \mathcal{P}, j \in \{1, \ldots, N\}$, and $\mathbb{I}_j(x) := \{t \in \mathbb{R} \,|\, x + t\mathbf{e}_j \in \mathcal{P}\}$ let

$$f_j^*(u, v) := \min_{t \in \mathbb{I}_j(v)} \|Au - A(v + t\,\mathbf{e}_j)\|_2^2.$$

**A Key Condition** (for a given matrix $A$ and a closed convex set $\mathcal{P}$)
For any $0 \neq u \in \Sigma_k \cap \mathcal{P}$, any index set $\mathcal{S} \subset \text{supp}(u)$ and an arbitrary optimal solution $v$ of $\min_{z \in \mathcal{P},\, \text{supp}(z) \subseteq \mathcal{S}} \|A(u - z)\|_2^2$, the following holds:

$$\mathbf{H}: \min_{j \in \text{supp}(u) \setminus \mathcal{S}} f_j^*(u, v) < \min_{j \in [\text{supp}(u)]^c} f_j^*(u, v).$$

**Theorem**

*Given a matrix $A \in \mathbb{R}^{m \times N}$ and a constraint set $\mathcal{P}$, suppose condition $(\mathbf{H})$ holds. Then the exact support recovery is achieved on $\Sigma_k \cap \mathcal{P}$.*

## Remarks on Condition H

It is computationally costly.

It does not only depend on $A$ but also critically relies on constraint set $\mathcal{P}$:

### Example

Let $d = (d_1, \ldots, d_N)^T \in \mathbb{R}^N$ with $d_i \neq 0$ for each $i$. Consider the set $\mathcal{P} := \{x \in \mathbb{R}^N \mid d^T x = 0\}$. For any $u, v \in \mathcal{P}$ and any index $j$, $f_j^*(u, v) = \|A(u - v)\|_2^2$ for any matrix $A$. Thus, for $x^* \in \Sigma_k \cap \mathcal{P}$, we have $\text{Argmin}_{j \in \{1,\ldots,N\}} f^*(x^*, 0) = \{1, \ldots, N\}$, so it is possible that $j_1^* \notin \text{supp}(x^*)$. Hence, **no matrix $A$ can achieve exact support recovery on the set $\mathcal{P}$.**

## Motivations to Identify a Practical Class of Sets

(i) Each set in this class contains sufficiently many sparse vectors.

(ii) It includes important sets arising from applications like $\mathbb{R}^N$ and $\mathbb{R}^N_+$.

(iii) Relatively easily verifiable sufficient recovery conditions can be established using general properties of this class of sets.

# Coordinate-Projection Admissible Sets

**Definition**

A set $\mathcal{P} \subseteq \mathbb{R}^N$ is coordinate-projection (CP) admissible if for every $u \in \mathcal{P}$ and $\mathcal{S} \subset \text{supp}(u)$, the vector $(u_{\mathcal{S}}; 0) \in \mathcal{P}$.

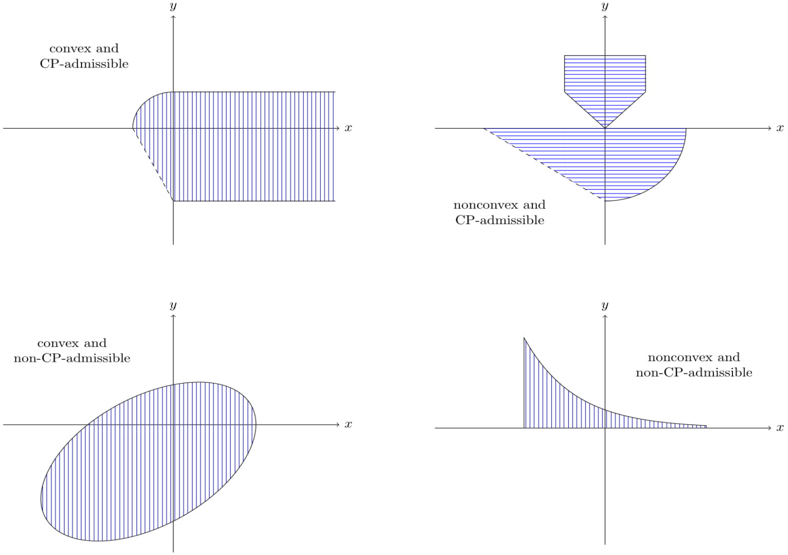
## Some Important Examples

$\mathbb{R}^N, \mathbb{R}^N_+,$ unit $\ell_p$-ball with $p \geq 0$, and more.

## Some Interesting Properties

1. A set $\mathcal{C} \subseteq \mathbb{R}^N$ is a CP admissible closed convex cone **if and only if** $\mathcal{C} = \mathbb{R}_{\mathcal{I}_1} \times (\mathbb{R}_+)_{\mathcal{I}_+} \times (\mathbb{R}_-)_{\mathcal{I}_-} \times \{0\}_{\mathcal{I}_0}$ such that $\mathcal{I}_1, \mathcal{I}_+, \mathcal{I}_,$ and $\mathcal{I}_0$ construct a partition of $\{1, \ldots, N\}$. Hence, the conic hull of a closed CP admissible set $\mathcal{P}$ is of form $\text{cone}(\mathcal{P}) = \mathbb{R}_{\mathcal{I}_1} \times (\mathbb{R}_+)_{\mathcal{I}_+} \times (\mathbb{R}_-)_{\mathcal{I}_-} \times \{0\}_{\mathcal{I}_0}$.
2. A set $\mathcal{P} \subseteq \mathbb{R}^N$ is CP admissible, closed and convex **if and only if** $\mathcal{P} = \mathcal{W} + \mathcal{C}$ where $\mathcal{W}$ is a CP admissible convex and compact set, and $\mathcal{C}$ is a CP admissible closed convex cone. (similar to Minkowski-Weyl)

# Examples to Clarify Geometry of CP-admissible Sets



# Generalization of Restricted Isometry and Orthogonality Constants

## Definition

For a given (possibly non-CP admissible) set $\mathcal{P}$, a matrix $A \in \mathbb{R}^{m\times N}$, and disjoint index sets $\mathcal{I}_1, \mathcal{I}_+, \mathcal{I}_-$ whose union is $\{1, \ldots, N\}$, we say that

• A real number $\delta$ is *of Property RI on* $\mathcal{P}$ if $(1-\delta)\|u-v\|_2^2 \leq \|A(u-v)\|_2^2$ for all $u, v \in \Sigma_k \cap \mathcal{P}$ with $\text{supp}(v) \subset \text{supp}(u)$, and $0 < \delta < 1$.

• A real number $\theta$ is *of Property RO on* $\mathcal{P}$ *corresponding to* $\mathcal{I}_1, \mathcal{I}_+, \mathcal{I}_-$ if $\theta > 0$ and for all $u, v \in \Sigma_K \cap \mathcal{P}$ with $\text{supp}(v) \subset \text{supp}(u)$, we have

$$\max\Big( \max_{j\in[\text{supp}(u)]^c\cap\mathcal{I}_1} |\langle A(u-v), A_{\bullet j}\rangle|,\ \max_{j\in[\text{supp}(u)]^c\cap\mathcal{I}_+} \langle A(u-v), A_{\bullet j}\rangle_+,\ \max_{j\in[\text{supp}(u)]^c\cap\mathcal{I}_-} \langle A(u-v), A_{\bullet j}\rangle_- \Big) \leq \theta \cdot \|u-v\|_2.$$

We also denote these two constants by $\delta_{k,\mathcal{P}}$ and $\theta_{k,\mathcal{P}}$ respectively.

# Uniform Recovery Condition

## Theorem

*Let $A \in \mathbb{R}^{m\times N}$ be a matrix with unit columns, and $\mathcal{P}$ be an irreducible, closed, convex, and CP admissible set in $\mathbb{R}^N$ whose conic hull is given by cone$(\mathcal{P}) = \mathbb{R}_{\mathcal{I}_1} \times (\mathbb{R}_+)_{\mathcal{I}_+} \times (\mathbb{R}_-)_{\mathcal{I}_-}$, where $\mathcal{I}_1, \mathcal{I}_+$ and $\mathcal{I}_-$ form a disjoint union of $\{1,\ldots,N\}$. Then condition $(\mathbf{H})$ holds on $\mathcal{P}$ if*

*(i) There exist constants $\delta_{k,\mathcal{P}}$ of Property RI and $\theta_{k,\mathcal{P}}$ of Property RO corresponding to $\mathcal{I}_1, \mathcal{I}_+$ and $\mathcal{I}_-$ such that $1-\delta_{k,\mathcal{P}} > \sqrt{k}\cdot\theta_{k,\mathcal{P}}$; or*

*(ii) There exist constants $\delta_{k,\text{cone}(\mathcal{P})}$ of Property RI and $\theta_{k,\text{cone}(\mathcal{P})}$ of Property RO corresponding to $\mathcal{I}_1, \mathcal{I}_+$ and $\mathcal{I}_-$ such that $1-\delta_{k,\text{cone}(\mathcal{P})} > \sqrt{k}\cdot\theta_{k,\text{cone}(\mathcal{P})}$.*

## Remark

Condition (ii) is easier to check due to simple structure of cone$(\mathcal{P})$.

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# Thank You For Your Attention!