Special classes of P-matrices in the interval setting

Matyáš Lorenc

Faculty of Mathematics and Physics, Charles university

24th May, 2021

1. Introduction
2. B-matrices
3. Doubly B-matrices
4. B_{π}^{R}-matrices
5. Generation
6. Conclusion

1. Introduction
2. B-matrices
3. Doubly B-matrices
4. B_π^R-matrices
5. Generation
6. Conclusion

Notation and useful matrix classes

Notation and useful matrix classes

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{IR}$

Notation and useful matrix classes

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{IR}$
- $\mathbb{F}^+$, $\mathbb{F}_0^+$

Notation and useful matrix classes

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{IR}$
- $\mathbb{F}^+$, $\mathbb{F}_0^+$
- $\mathbb{F}^{m\times n}$, $\mathbb{F}^n$

Notation and useful matrix classes

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{IR}$
- $\mathbb{F}^+$, $\mathbb{F}_0^+$
- $\mathbb{F}^{m\times n}$, $\mathbb{F}^n$
- Let $n \in \mathbb{N}$, then $[n] = \{1, 2, \ldots, n\}$.

Notation and useful matrix classes

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{IR}$
- $\mathbb{F}^+$, $\mathbb{F}_0^+$
- $\mathbb{F}^{m\times n}$, $\mathbb{F}^n$
- Let $n \in \mathbb{N}$, then $[n] = \{1, 2, \dots, n\}$.
- Let $A \in \mathbb{F}^{n\times n}$. Then $\forall i \in [n]:\ r_i^+ = \max\{0, a_{ij} | j \neq i\}$.

Notation and useful matrix classes

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{IR}$
- $\mathbb{F}^+$, $\mathbb{F}_0^+$
- $\mathbb{F}^{m\times n}$, $\mathbb{F}^n$
- Let $n \in \mathbb{N}$, then $[n] = \{1, 2, \dots, n\}$.
- Let $A \in \mathbb{F}^{n\times n}$. Then $\forall i \in [n]: \; r_i^+ = \max\{0, a_{ij} | j \neq i\}$.

Definition 1.1 (Z-matrix)

Let $A \in \mathbb{R}^{n\times n}$. We say that A is a Z-matrix, if all its off-diagonal elements are non-positive.

Notation and useful matrix classes

Definition 1.2 (circulant matrix)

Let $A \in \mathbb{R}^{n\times n}$. We say that A is a circulant matrix, if all its rows are each cyclic permutations of the first row with offset equal to the row index minus one, hence if it takes the following form:

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \ddots & & c_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & & \ddots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

P-matrices

Definition 1.3 (P-matrix)

Let $A \in \mathbb{R}^{n \times n}$. We say that A is a P-matrix, if all its principal minors are positive.

P-matrices

Definition 1.3 (P-matrix)

Let $A \in \mathbb{R}^{n\times n}$. We say that A is a P-matrix, if all its principal minors are positive.

Definition 1.4 (Linear complementarity problem)

Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Then the linear complementarity problem, denoted $LCP(M, q)$, is a task to find a vector z which satisfies the following:

- $z \geq 0$
- $Mz + q \geq 0$
- $z^T(Mz + q) = 0$

Interval analysis

Interval analysis

Definition 1.5 (interval matrix)

An interval matrix $\boldsymbol{A}$*, denoted by* $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$*, is defined as*

$$\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] = \left\{A \in \mathbb{R}^{m\times n} \middle| \underline{A} \leq A \leq \overline{A}\right\},$$

where $\underline{A}, \overline{A}$ *are called lower, respectively upper bound matrices of* $\boldsymbol{A}$*.*
We can as well look at $\boldsymbol{A}$ *as matrix, which has entries from* $\mathbb{IR}$*, hence*
$\forall i \in [m], \forall j \in [n] : \boldsymbol{a}_{ij} = \left[\underline{a_{ij}}, \overline{a_{ij}}\right]$.
If we define matrices $A^C = \frac{1}{2}\left(\overline{A} + \underline{A}\right)$ *and* $A^\Delta = \frac{1}{2}\left(\overline{A} - \underline{A}\right)$*, then we can define* $\boldsymbol{A}$ *alternatively as*

$$\boldsymbol{A} = \left[A^C \pm A^\Delta\right] = \left[A^C - A^\Delta, A^C + A^\Delta\right].$$

1 Introduction

2 B-matrices

3 Doubly B-matrices

4 B_π^R-matrices

5 Generation

6 Conclusion

Real B-matrices

Definition 2.1 (B-matrix)

Let $A \in \mathbb{R}^{n \times n}$. Then we say that A is a B-matrix, if $\forall i \in [n]$ the following holds:

$$a) \quad \sum_{j=1}^{n} a_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \frac{1}{n} \sum_{j=1}^{n} a_{ij} > a_{ik}$$

Real B-matrices

Definition 2.1 (B-matrix)

Let $A \in \mathbb{R}^{n\times n}$. Then we say that A is a B-matrix, if $\forall i \in [n]$ the following holds:

$$a) \quad \sum_{j=1}^{n} a_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \frac{1}{n} \sum_{j=1}^{n} a_{ij} > a_{ik}$$

$$\sum_{j=1}^{n} a_{ij} > n \cdot r_i^+$$

Real B-matrices

Definition 2.1 (B-matrix)

Let $A \in \mathbb{R}^{n\times n}$. Then we say that A is a B-matrix, if $\forall i \in [n]$ the following holds:

$$a) \quad \sum_{j=1}^{n} a_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \frac{1}{n}\sum_{j=1}^{n} a_{ij} > a_{ik}$$

$\sum_{j=1}^{n} a_{ij} > n \cdot r_i^+$	$a_{ii} - r_i^+ > \sum_{j\neq i} \left(r_i^+ - a_{ij}\right)$

Interval B-matrices

Interval B-matrices

Theorem 2.2

Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following two properties hold:

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \sum_{j \neq k} \underline{a}_{ij} > (n-1) \cdot \overline{a}_{ik}$$

Interval B-matrices

Theorem 2.2

Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$. Then $\mathbf{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following two properties hold:

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \sum_{j \neq k} \underline{a}_{ij} > (n-1) \cdot \overline{a}_{ik}$$

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0 \qquad\qquad b) \quad \forall k \in [n] \setminus \{i\} : \quad \underline{a}_{ii} - \overline{a}_{ik} > \sum_{\substack{j \neq i \\ j \neq k}} \left(\overline{a}_{ik} - \underline{a}_{ij} \right)$$

Interval B-matrices - Characterization through reduction

Interval B-matrices - Characterization through reduction

Proposition 2.3

Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ and let A_i be matrices defined as follows:

$$A_i = (a_{m_1 m_2}); \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } m_1 \neq i, m_2 = i, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n] : A_i$ is a B-matrix.

Interval B-matrices - Z-matrices

Theorem 2.4

Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ be an interval Z-matrix. Then the following is equivalent:

1. $\mathbf{A}$ *is an interval B-matrix,*
2. $\forall i \in [n]: \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0,$
3. $\forall i \in [n]: \quad \underline{a}_{ii} > \sum_{j\neq i} |\underline{a}_{ij}|.$
4. $\underline{A}$ *is a B-matrix.*

Interval B-matrices - Closure properties

Interval B-matrices - Closure properties

Proposition 2.5

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval B-matrix and $\boldsymbol{\alpha} \in \mathbb{IR}^{+}$ be an interval, such that:

$$\underline{\alpha}/\overline{\alpha} \quad > \quad \max\left(\left\{\frac{\sum\limits_{j:a_{ij}<0} -\underline{a}_{ij}}{\sum\limits_{j:a_{ij}>0} \underline{a}_{ij}}\,\middle|\, i \in [n]\right\} \cup \left\{\frac{\sum\limits_{\substack{j:\underline{a}_{ij}<0\\ j\neq k}} -\underline{a}_{ij} + (n-1)\cdot \overline{a}_{ik}}{\sum\limits_{\substack{j:\underline{a}_{ij}>0\\ j\neq k}} \underline{a}_{ij}}\,\middle|\, i \in [n], k \in [n]\setminus\{i\} : \overline{a}_{ik} > 0\right\}\right).$$

Then matrix $\boldsymbol{\alpha}\cdot\boldsymbol{A}$ is also an interval B-matrix.

Interval B-matrices - Closure properties

Proposition 2.6

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval B-matrix and $\boldsymbol{\alpha} = \left[\alpha^C \pm \alpha^\Delta\right] \in \mathbb{IR}^+$, such that:

$$\alpha^\Delta < \min\left(\left\{\frac{\alpha^C \cdot \sum_{j=1}^{n} \underline{a}_{ij}}{\sum_{j=1}^{n} |\underline{a}_{ij}|} \,\middle|\, i \in [n]\right\} \cup \left\{\frac{\alpha^C \cdot \left(\sum_{j\neq k} \underline{a}_{ij} - (n-1)\cdot \overline{a}_{ik}\right)}{\sum_{j\neq k} |\underline{a}_{ij}| + (n-1)\cdot |\overline{a}_{ik}|} \,\middle|\, i \in [n], k \in [n]\setminus\{i\}\right\}\right).$$

Then matrix $\boldsymbol{\alpha} \cdot \boldsymbol{A}$ is also an interval B-matrix.

1 Introduction

2 B-matrices

3 Doubly B-matrices

4 B_π^R-matrices

5 Generation

6 Conclusion

Real doubly B-matrices

Definition 3.1 (doubly B-matrix)

Let $A \in \mathbb{R}^{n\times n}$. Then we say that A is a doubly B-matrix, if $\forall i \in [n]$ the following holds:

a) $a_{ii} > r_i^+$

b) $\forall j \in [n] \setminus \{i\} : \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \left(\sum_{k\neq i}\left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k\neq j}\left(r_j^+ - a_{jk}\right)\right)$

Circulant matrices

Theorem 3.2

Let $A \in \mathbb{R}^{n\times n}$ be a circulant matrix. Then the following are equivalent:

1. *A is a B-matrix.*
2. *A is a doubly B-matrix.*
3. $a_{11} - r_1^+ > \sum_{j\neq 1} \left(r_1^+ - a_{1j}\right)$

Interval doubly B-matrices

Interval doubly B-matrices

Theorem 3.3

Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if the following two properties holds:

a) $\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$ *and*

b) $\forall i, j \in [n], j \neq i, \forall k, l \in [n], k \neq i, l \neq j:$

$$\vdots$$

Interval doubly B-matrices

a) $\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$ and

b) $\forall i, j \in [n], j \neq i, \forall k, l \in [n], k \neq i, l \neq j:$

1. $$(\underline{a}_{ii} - \overline{a}_{ik})(\underline{a}_{jj} - \overline{a}_{jl}) > \left(\max\left\{0, \sum_{\substack{m \neq i \\ m \neq k}} (\overline{a}_{ik} - \underline{a}_{im})\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}} (\overline{a}_{jl} - \underline{a}_{jm})\right\}\right)$$

2. $$\underline{a}_{ii}(\underline{a}_{jj} - \overline{a}_{jl}) > \left(\max\left\{0, -\sum_{m \neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}} (\overline{a}_{jl} - \underline{a}_{jm})\right\}\right)$$

3. $$\underline{a}_{ii} \cdot \underline{a}_{jj} > \left(\max\left\{0, -\sum_{m \neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m \neq j} \underline{a}_{jm}\right\}\right)$$

Interval doubly B-matrices - Characterization through reduction

Interval doubly B-matrices - Characterization through reduction

Proposition 3.4

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *for* $n \geq 4$ *and let us define* $A_{(i,k),(j,l)} \in \mathbb{R}^{n\times n}$ *as follows:*

$$A_{(i,k),(j,l)} = (a_{m_1 m_2})\,; \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1, m_2) = (i, k), \\ \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ *is an interval doubly B-matrix if and only if* $\forall i, j \in [n], j > i, \forall k, l \in [n], k \neq i, l \neq j : A_{(i,k),(j,l)}$ *is a doubly B-matrix.*

Interval doubly B-matrices - Characterization through reduction

Proposition 3.5

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and let us define $A_{(i,k),(*,l)}$ and ${}_iA_{(*,l)} \in \mathbb{R}^{n\times n}$ as follows:

$$A_{(i,k),(*,l)} = (a_{m_1 m_2}); \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1, m_2) = (i,k), \\ \overline{a}_{m_1 l} & \text{if } m_2 = l \wedge m_1 \neq i \wedge m_1 \neq l, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

$${}_iA_{(*,l)} = (a'_{m_1 m_2}); \quad a'_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 l} & \text{if } m_2 = l \wedge m_1 \neq i \wedge m_1 \neq l, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if $\forall i, l \in [n] : ({}_iA_{(*,l)}$ is a doubly B-matrix $\quad \wedge \quad \forall k \in [n] \setminus \{i\} : A_{(i,k),(*,l)}$ is a doubly B-matrix$)$.

Interval doubly B-matrices - Sufficient conditions

Theorem 3.6

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ interval Z-matrix. Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if $\underline{A}$ is a doubly B-matrix.

Theorem 3.7

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}, \forall i \in [n] : k_i \in \operatorname{argmax}\{\overline{a}_{ij} | j \neq i\}$ and $\forall i \in [n] : k_i' \in \operatorname{argmax}\{\underline{a}_{ij} | j \neq i\}$. Let us define $\widetilde{A} \in \mathbb{IR}^{n\times n}$ as follows:

$$\widetilde{A} = (\widetilde{a}_{m_1 m_2})\,; \quad \widetilde{a}_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 k_{m_1}} & \text{if } m_2 = k_{m_1}, \\ \underline{a}_{m_1 m_1} & \text{if } m_2 = m_1, \\ \min\{\underline{a}_{m_1 m_2}, \underline{a}_{m_1 k_{m_1}}\} & \text{otherwise.} \end{cases}$$

If $\forall i \in [n] : \underline{a}_{ik_i'} \geq 0$ and $\widetilde{A}$ is a doubly B-matrix, then $\boldsymbol{A}$ is an interval doubly B-matrix.

Interval "circulant" matrices

Theorem 3.8

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ such that $\underline{A}$ and $\overline{A}$ are circulant. Then the following is equivalent:

1. *$\boldsymbol{A}$ is an interval doubly B-matrix*
2. *$\boldsymbol{A}$ is an interval B-matrix*
3. *It holds that*

$$a)\quad \underline{a}_{11} > -\sum_{j\neq 1} \underline{a}_{1j}$$

$$b)\quad \forall k \in [n] \setminus \{1\} :\ \underline{a}_{11} - \overline{a}_{1k} > \sum_{\substack{j\neq 1\\ j\neq k}} (\overline{a}_{1k} - \underline{a}_{1j})$$

1. Introduction
2. B-matrices
3. Doubly B-matrices
4. B_π^R-matrices
5. Generation
6. Conclusion

Real B_π^R-matrices

Definition 4.1 (B_π^R-matrix)

Let $A \in \mathbb{R}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $R \in \mathbb{R}^n$ be a vector formed by the row sums of A (hence $\forall i \in [n]: \; R_i = \sum_{j=1}^n a_{ij}$). Then we say that A is a B_π^R-matrix, if $\forall i \in [n]$:

$$\begin{aligned} &a) \quad R_i > 0 \\ &b) \quad \forall k \in [n] \setminus \{i\} : \quad \pi_k \cdot R_i > a_{ik} \end{aligned}$$

Real B_π^R-matrices

Proposition 4.2

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix with positive row sums and let $R \in \mathbb{R}^n$ be a vector formed by the row sums of A (hence $\forall i \in [n] : \ R_i = \sum_{j=1}^n a_{ij} > 0$). Then there exists a vector $\pi \in \mathbb{R}^n$ satisfying $0 < \sum_{j=1}^n \pi_j \leq 1$ such that A is a B_π^R-matrix if and only if

$$\sum_{j=1}^n \max\left\{ \frac{a_{ij}}{R_i} \,\middle|\, i \neq j \right\} < 1.$$

Interval B_π^R-matrices

Interval B_π^R-matrices

Definition 4.3 (homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix)

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$. Then we say that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix, if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}$ such that A is a (real) B_π^R-matrix.

Interval B_π^R-matrices

Definition 4.3 (homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix)

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$. Then we say that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix, if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}$ such that A is a (real) B_π^R-matrix.

Definition 4.4 ((heterogeneous) interval $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrix)

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{R} \in \mathbb{IR}^n$. Then we say that $\boldsymbol{A}$ is a (heterogeneous) interval $B_\Pi^{\boldsymbol{R}}$-matrix, if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}, \exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$: A is a (real) B_π^R-matrix.

Homogeneous interval B_π^R-matrices

Homogeneous interval B_π^R-matrices

Theorem 4.5

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$ be a vector of intervals of the individual row sums in matrix $\boldsymbol{A}$. Then $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n]$ the following properties hold:

$$
\begin{aligned}
a)&\quad \underline{R}_i > 0\\
b)&\quad \forall k \in [n] \setminus \{i\} :\\
&\quad \vdots
\end{aligned}
$$

Homogeneous interval B_π^R-matrices

a) $\underline{R}_i > 0$

b) $\forall k \in [n] \setminus \{i\}$:

$$\left(\pi_k > 1 \quad \Rightarrow \quad \sum_{j \neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \underline{a}_{ik}\right) \quad \wedge$$

$$\wedge \quad \left(0 < \pi_k \leq 1 \quad \Rightarrow \quad \sum_{j \neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}\right) \quad \wedge$$

$$\wedge \quad \left(\pi_k = 0 \quad \Rightarrow \quad 0 > \overline{a}_{ik}\right) \quad \wedge$$

$$\wedge \quad \left(\pi_k < 0 \quad \Rightarrow \quad \sum_{j \neq k} \overline{a}_{ij} < \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}\right)$$

Homogeneous interval B_π^R-matrices

Theorem 4.6

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval square matrix with positive row sums intervals (hence $\forall i \in [n] : \sum_{j=1}^n \underline{a}_{ij} > 0$). Then there exists a vector $\pi \in \mathbb{R}^n$ satisfying $0 < \sum_{j=1}^n \pi_j \leq 1$ such that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if

$$\sum_{j=1}^n \max \left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < 1.$$

Interval B_Π^R-matrix (heterogeneous)

Interval B_Π^R-matrix (heterogeneous)

Theorem 4.7

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval square matrix with positive row sums intervals. Then $\boldsymbol{A}$ is an interval B_Π^R-matrix if and only if $\exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and that $\boldsymbol{A}$ is a homogeneous interval B_π^R-matrix.

Interval B_Π^R-matrix (heterogeneous)

Theorem 4.7

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval square matrix with positive row sums intervals. Then $\boldsymbol{A}$ is an interval B_Π^R-matrix if and only if $\exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and that $\boldsymbol{A}$ is a homogeneous interval B_π^R-matrix.

Definition 4.8 (interval B_π^R-matrix)

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$. Then we say that $\boldsymbol{A}$ is an interval B_π^R-matrix if it is a homogeneous interval B_π^R-matrix.

Interval B_π^R-matrices - Characterization through reduction

Interval B_π^R-matrices - Characterization through reduction

Proposition 4.9

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$ be a vector of intervals of the individual row sums in matrix $\boldsymbol{A}$. Let $\forall i \in [n] : A_i \in \mathbb{R}^{n\times n}$ defined as follows:

1. *if $\pi_i > 1$, then: $A_i = \underline{A}$*
2. *else if $0 \leq \pi_i \leq 1$, then: $A_i = (a_{m_1 m_2})$;* $a_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } m_1 \neq i, m_2 = i, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$
3. *else if $\pi_i < 0$, then: $A_i = (a_{m_1 m_2})$;* $a_{m_1 m_2} = \begin{cases} \underline{a}_{m_1 m_2} & \text{if } m_1 = i, \\ \overline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$

Then $\boldsymbol{A}$ is an interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n]$: A_i is a B_π^R-matrix, where $R \in \mathbb{R}^n$ is a vector of values corresponding to the row sums of A_i.

1 Introduction

2 B-matrices

3 Doubly B-matrices

4 B_π^R-matrices

5 Generation

6 Conclusion

B-matrices

B-matrices

Let $n \in \mathbb{N}$. Let $N, N' \in \mathbb{R}$, $N \geq n$, $N' \geq 2n-1$ arbitrary and let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ defined as follows:

$$\boldsymbol{A} = (\boldsymbol{a}_{ij}); \quad \boldsymbol{a}_{ij} = \begin{cases} [2n-1, N'] & \text{if } i = j, \\ [-1, 1] & \text{if } i \neq j. \end{cases}$$

$$\boldsymbol{A}' = (\boldsymbol{a}'_{ij}); \quad \boldsymbol{a}'_{ij} = \begin{cases} [n, N] & \text{if } i = j, \\ [-1, \frac{1}{n-1}] & \text{if } i \neq j. \end{cases}$$

$$\boldsymbol{A}'' = (\boldsymbol{a}''_{ij}); \quad \boldsymbol{a}''_{ij} = \begin{cases} [n, N] & \text{if } i = j, \\ [\frac{-1}{n-1}, 1] & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}$, $\boldsymbol{A}'$ and $\boldsymbol{A}''$ are interval B-matrices.

Doubly B-matrices

Doubly B-matrices

$$\max \sum_{m=1}^{n} \overline{x}_m - \sum_{m=1}^{n} \underline{x}_m;$$

$$\overline{x}_k \le \underline{a}_{ii} - \overline{a}_{ik} \qquad \text{for } k \ne i$$

$$-\sum_{m\ne i} \underline{x}_m \le \frac{\underline{a}_{ii}\cdot \underline{a}_{jj}}{-\sum_{m\ne j} \underline{a}_{jm}} + \sum_{m\ne i} \underline{a}_{im} \qquad \text{for } j \ne i : -\sum_{m\ne j} \underline{a}_{jm} > 0$$

$$-\sum_{m\ne i} \underline{x}_m \le \frac{\underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right)}{\sum_{\substack{m\ne j\\ m\ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)} + \sum_{m\ne i} \underline{a}_{im} \qquad \text{for } j \ne i, l \ne j : \sum_{\substack{m\ne j\\ m\ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right) > 0$$

$$\left(\frac{\underline{a}_{jj}}{-\sum_{m\ne j} \underline{a}_{jm}} + (n-2)\right)\cdot \overline{x}_k - \sum_{\substack{m\ne i\\ m\ne k}} \underline{x}_m \le \frac{\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\underline{a}_{jj}}{-\sum_{m\ne j} \underline{a}_{jm}} - \sum_{\substack{m\ne i\\ m\ne k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right) \qquad \text{for } k \ne i, j \ne i : -\sum_{m\ne j} \underline{a}_{jm} > 0$$

$$\left(\frac{\left(\underline{a}_{jj} - \overline{a}_{jl}\right)}{\sum_{\substack{m\ne j\\ m\ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)} + (n-2)\right)\cdot \overline{x}_k - \sum_{\substack{m\ne i\\ m\ne k}} \underline{x}_m \le \frac{\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right)}{\sum_{\substack{m\ne j\\ m\ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)} - \sum_{\substack{m\ne i\\ m\ne k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right) \qquad \text{for } k \ne i, j \ne i, l \ne j : \sum_{\substack{m\ne j\\ m\ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right) > 0$$

$$\overline{x}_i = 0, \underline{x}_i = 0, \overline{x} \ge 0, \underline{x} \le 0$$

1 Introduction

2 B-matrices

3 Doubly B-matrices

4 B_π^R-matrices

5 Generation

6 Conclusion

Open problems

Definition 6.1 (Parametric interval matrix)

Let $k, m, n \in \mathbb{N}$, $\boldsymbol{p} \in \mathbb{IR}^k$ and $\{A_0, A_1, \ldots, A_k\} \subset \mathbb{R}^{m \times n}$. Then we define parametric interval matrix $\boldsymbol{A}(\boldsymbol{p})$ as follows:

$$\boldsymbol{A}(\boldsymbol{p}) = A_0 + \sum_{i=1}^{k} \boldsymbol{p}_i A_i$$

Open problems

Definition 6.2 (Mime)

*Let $A \in \mathbb{R}^{n\times n}$. Then we call A a mime, which stands for **M**-matrix and **I**nverse **M**-matrix **E**xtension, if for some $s_1, s_2 \in \mathbb{R}$, $P_1, P_2 \in {\mathbb{R}_0^+}^{n\times n}$, such that $\exists u \in {\mathbb{R}_0^+}^n$ which satisfies*

$$P_1 u < s_1 u \text{ and } P_2 u < s_2 u,$$

it takes the form of

$$A = (s_1 I_n - P_1)(s_2 I_n - P_2)^{-1}.$$

- $s_2 = 1, P_2 = 0 \quad \rightarrow \quad$ M-matrices
- $s_1 = 1, P_1 = 0 \quad \rightarrow \quad$ inverse M-matrices

Theorem 6.3 ("The end is coming" theorem)

1. *This is the end.*
2. *Everyone is already asleep by now.*

Proof.

1. Trivial.
2. Look around. (If you are not sleeping, then sweet dreams.)

$\boxtimes$