Towards understanding winter road maintenance

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Let G^s denote the symmetric orientation of graph G, i.e., each edge of G appears twice oppositely oriented in G^s. The solution has the following features.

- We construct a partition $P = \{P_1, \ldots, P_r\}$ of the set of arcs of G^s into sets $P_1, \ldots, P_r$ and for each i we assign vertex (depot) $d_i \in D$. We assume the type of maintenance m constant in P_i. We also assume that the oppositely oriented edges belong to the same P_i.
- We construct, for each i, set R_i so that $P_i \subset R_i$ and each arc of P_i may be reached from d_i by a directed closed walk of R_i.
- For each i, we design a schedule of maintenance of the edges of P_i by a single vehicle starting and terminating at d_i and using only arcs of R_i, with a fixed time period.

The Route Requirements

- Length of working shift: 8 hours in the Czech Republic, plus multiple safety breaks.
  For simplicity, we will assume in our model that the working shift lasts a fixed amount of time, e.g., six hours, uninterrapted.
- The road network has a service hierarchy which partitions the roads into classes.
  In the Czech Republic, there are three such classes: Arterial roads priority (1). Priority (2) is assigned to bus routes and other important routs. Each class is associated with maximum time of service completion.
- The capacity c_m describing the maximum length of a route which can be maintained with only one loading of the material $m \in M$ (Then back to the depot).
- It turns out that it is more convenient to define c_m as a fraction of the maximum route-length of one vehicle rather than as a 'number'.

- Maintaining plan is a tuple (G, P, d, α, z, p). $G = (V, E)$ a graph representing a *road network*. Vertices represent crossroads (and dead ends) and edges represent roads among them.
- $\alpha : E \to R^+$ gives to every edge a non-negative length,
- Let $z \geq 1$ denote the number of priority classes of roads
- $p : E \to \{1, \ldots, z\}$ priority level,
- $P \subset E$ is the set of maintained edges. We will assume $P = E$ (It is practical)
- $d \in V$ is called the *depot*

Vehicle route of the maintaining plan

- A vehicle route is sequence of arcs of E^S, $w = (e_1, \ldots, e_\ell)$, which starts and ends at d,
- Given numbers L,c and functions $t \colon \{1, \ldots, z\} \times Z^+ \to R$ and $f : E \to Z^+$
- Each arc of E^S is traversed at least once by w and at most $f(e)$ times,
- Total length limit L: $\sum_{q \leq \ell} \alpha(e_q) \leq L$,
- For all pairs $i < j$ (taken cyclically) and y such that
  [i] e_i appears exactly $(y-1)$ times among $e_1, \ldots, e_{i-1}$,
  [ii] $e_i = e_j$ and
  [iii] $e_i \neq e_k$ for $i < k < j$,
  we have
  $\sum_{q \geq i}^{j-1} \alpha(e_q) \leq Lt(p(e_i), y))$,
- A trip is a segment T_j of route w which starts and ends by depot d and the interior of T_j (ending vertices are excluded) does not contain the depot. A vehicle route is thus a sequence of trips.
- capacity c: for all trips $T_j, \sum_{e \in T_j \cap P^S} \alpha(e) \leq cL$.

Routes hard

Deciding existence of a vehicle route is hard even for stars and all associated parameters and functions except α trivial (reduction to number partitioning problem).

Features

- Why the above hardness proof is relevant? Lengths are bounded, degrees are bounded...
- **Weighted** versus **unweighted** version:
  The function α on edges should be interpreted as **cost** of traversing the road segment rather than the length of the segment.
- For some roads, the cost is not linear in length (mountain roads ...)
- Public satisfaction/complaints are **important**
- Routes are very expensive (vehicles, gas, drivers)

Main Contribution for single routes

- Existence of a vehicle route visiting each arc at most F times is **efficient** for unweighted trees of bounded degree (and for bounded tree-width graphs)
- This is the basis of our algorithm for the winter road maintenance
- Both restrictions (F and bounded degree) are necessary
- A simplification leads to an interesting **graph cutting problem**
- A (first as far as we know) attempt to formalise public satisfaction

Dynamic Programming argument

Theorem

Fixed integers F, Δ. There exists a polynomial time algorithm which for a tree $T = (V, E)$ rooted in d with maximal degree at most Δ, function $f : E \to N$ such that $f(e) \leq F$ for all $e \in E$ and function $g : E \times \{1, \ldots, F\} \to N$ decides whether there exists a closed walk w starting at r satisfying

- *Every edge e of T^s is traversed $f(e)$-times in both directions.*
- *For every edge e of T^s and $y \leq f(e)$, there are at most $g(e, y)$ steps between $y-$th and $(y+1)-$st traverses of e, taken cyclically.*

Theorem

Fixed integers F, Δ, k. There exists a polynomial time algorithm which for a graph $G = (V, E)$ rooted in d and with maximal degree at most Δ, given along with its canonical tree decomposition (W, b) of width $k-1$ and functions $f : E^s \to N$ such that $f(e) \leq F$ for all $e \in E^s$ and $g : E^s \times \{1, \ldots, F\} \to N$ decides whether there exists a closed walk w starting at d satisfying

- *Every edge e is traversed $f(e)$-times in both directions.*
- *For every edge e and $y \leq f(e)$, there are at most $g(e, y)$ steps between the $y-$th and $(y+1)-$th traverses of e, taken cyclically.*

Theorem

It is NP-complete to decide whether a given binary tree $T = (V, E)$ rooted in d and function $f, g : E \to N$ there exists a closed walk w starting at r satisfying

- *Every edge e is traversed $f(e)$-times in both directions.*
- *For every edge e, there are at most $g(e)$ steps between two consecutive traverses of e in both direction, taken cyclically.*

The problem is NP-complete even if we restrict f to be non-increasing on all paths from the depot.

Theorem

Fixed integer F. It is NP-complete to decide whether a given tree $T = (V, E)$ rooted in d, function $f : E \to N$ such that $f(e) \leq F$ for every edge e and function $g : E \times \{1, \dots, F\} \to N$ there exists a closed walk w starting at d satisfying

- *Every edge e is traversed $f(e)$-times in both directions.*
- *For every edge e and $y \leq f(e)$, there are at most $g(e)$ steps between two $y-$the and $(y+1)-$st traverses of e in both direction, taken cyclically.*

- Graph Cutting Problem is to find, for a given graph G with root d and set of numbers $t_1, \ldots, t_k$, a cover of the edge-set of G by connected subgraphs $G_1, \ldots, G_k$ rooted in d of sizes $t_1, \ldots, t_k$.
- Let k be any fixed number. There is a polynomial algorithm to solve the Tree Cutting Problem. This solves polynomially vehicle routing for trees and constant priority.
- There is always a solution for $k = 2$ and G 2-connected.
- NP-complete in general for $k = 2$.

Public Complaints

- Services of winter road maintenance are mostly of higher quality than what legislation constraints require.
- The complaints from public against the winter road maintenance is focused on in media or in election campaigns.
- The experience is that residents make complaints about the winter road maintenance plans if they think that they are **treated in an unfair manner**, i.e., if their neighbourhood is 'skipped' in the service.
- We will formalise public complaints in graphs.
- We assume a fixed cyclic order $O(v)$ of neighbours of each vertex v. Such orders of incident roads are historically acknowledged in the communities.

Unfairness

We make a rational assumption that the number of public complaints can be deduced from the structure of a vehicle route and in particular from its perceived unfairness.
Let d be the depot of degree 1. Let $w = (e_1, ..., e_l)$ be a vehicle route.
For $e_i = (s_i, t_i)$

- let $(w, i)+$ denote the edge incident with t_i which follows (s_i, t_i) in $O(t_i)$
- let $(w, i)-$ denote the edge incident with s_i which precedes (s_i, t_i) in $O(s_i)$.
- If no orientation of $(w, i)+$ belongs to $(e_1, ..., e_{i+1})$ then we say that edge $(w, i)+$ has a forward complaint.
- If no orientation of $(w, i+1)-$ belongs to $(e_1, ..., e_{i+1})$ then we say that edge $(w, i+1)-$ has a backward complaint .
- The unfairness index of the route w, denoted by $Uf(w)$, is the sum of the number of edges which have a forward complaint and the number of edges which have a backward complaint.

Theorem (Noga Alon)

Every necklace with ka_i beads of color i, $1 \leq i \leq s$, has a k-splitting of size at most $(k-1) \cdot s$.

- Finding smallest number of cuts is NP-complete even for $k = 2$ and each $a_i = 1$ (Bonsman, Eppig and Hochstttler; Meunier)
- Can one find efficiently the guaranteed splitting ?
- This was finally answered negatively in 2019 by Filos-Ratsikas and Goldberg:
  finding necklace splitting guaranteed by Alon's theorem is PPA-complete even for $k = 2$.
  We say that problem is *PPA-complete* if it is polynomial time equivalent to the LEAF problem.
- An instance of the problem called *LEAF* consists of a graph G of maximum degree 2, whose 2^n vertices are represented by $0, 1$ sequences of length n; G is given by an algorithm that takes as input a vertex and outputs its neighbours. Moreover, the vertex 0 has degree 1. The goal is to output another vertex of degree 1.

Theorem

There exists a polynomial reduction of the necklace splitting problem to the unfairness minimisation problem where the network is a star rooted in its leaf, with edge-weights, each edge maintained at least once and $c = 1/k$.

Proof. Let $n = \sum_{l=1}^{s} ka_i$ be the length of the necklace.
Define numbers $M_1, \ldots, M_s$ so that $M_i > n \sum_{j<i} M_j$
Let T be a star with $n+1$ leaves $v_0, \ldots, v_n$ and center x. Let $d = v_0$ and let each edge xv_i has weight M_j if the bead in position i has color j. The weight of xv_0 is zero. Finally let L be twice the sum of all the edge-weights and let $c = 1/k$.

Weighted tree cutting with a guaranteed Unfairness is PPA-complete.

- Min necklace splitting hard for $k = 2$, and each $a_i = 1$. Relating this special case to the Unfairness:
- Is there a set of natural numbers $M_1, \ldots, M_s$ such that
- for each $i \in \{1, \ldots, s\}$, M_i bounded by a fixed power of s and
- all partial sums of M_i's are pairwise distinct.
- Let $f(n) = \min\{\max S : |S| = n \text{ and } S \text{ has distinct subset sums}\}$
- Paul Erdös conjectured in 1931 that for some constant c

$$f(n) \geq c2^n.$$

- Conway and Guy found a construction of sets with distinct subset sum, now called the *Conway-Guy sequence*. This was later improved by Lunnan and then by Bohman to $f(n) \leq 0.22002 \cdot 2^n$ (for n sufficiently large).
- The best known lower bound, up to the constant, has been proved by Erdös and Moser in 1955,
$f(n) \geq 2^n/(10\sqrt{n})$.

Conclusion

- Unfairness index polynomial for tree cutting, fixed k
- Is there Alon's theorem for trees, planar graphs? Trees, $k = a_1 = a_2 = 2$: might be also s.