Bilevel Programming with(out) Intervals

The Best, the Worst and the Corrected

Optimization Seminar (22. 3. 2021)

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Outline

Linear Bilevel Programming

Linear Bilevel Programming

$$\begin{aligned} \min_{x,y} \quad & c^T x + d^T y \\ \text{s. t.} \quad & A_1 x + B_1 y \geq b_1 \\ & y \in M(x) \end{aligned}$$

$$\text{where} \quad M(x) = \begin{aligned} \operatorname{argmin}_y \quad & a^T y \\ \text{s. t.} \quad & A_2 x + B_2 y \geq b_2 \end{aligned}$$

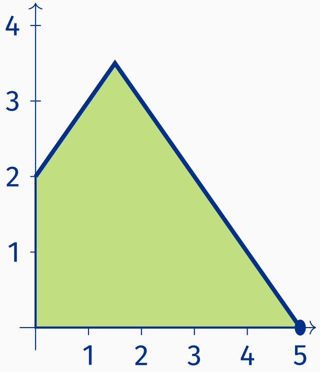
Constraint Region: $\{(x, y) : A_1 x + B_1 y \geq b_1, A_2 x + B_2 y \geq b_2\}$
Inducible Region: $\{(x, y) : A_1 x + B_1 y \geq b_1, y \in M(x)\}$

Linear Bilevel Programming: An Example

$$\min_{x,y} \; y$$

$$y \leq 3$$

$$y \in \operatorname{argmax}_y \left\{ y : \begin{aligned} x + y &\leq 5, \\ -x + y &\leq 2, \\ x, y &\geq 0 \end{aligned} \right\}$$



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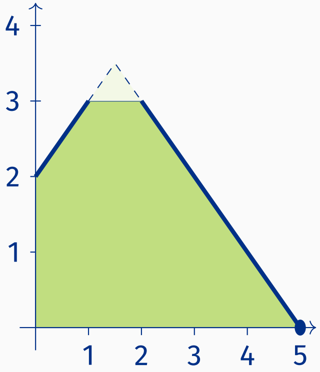


Linear Bilevel Programming: Complexity

Theorem (Jeroslow, 1985; Ben–Ayed & Blair, 1990)

Linear bilevel programming is NP-hard.

Theorem (Hansen et al., 1992)

Linear bilevel programming is strongly NP-hard.

Reduction from the KERNEL problem on graphs:

$$\begin{aligned} \max_x \min_y \Big\{ -\sum_j y_j : \\ x_j + x_k &\leq 1, && \forall j, k : (v_j, v_k) \in A, \\ y_j &\leq 1 - x_j, && \forall j \\ y_j &\leq 1 - x_k, && \forall j, k : (v_j, v_k) \in A, \\ x, y &\geq 0 \Big\} \end{aligned}$$

High-point Relaxation

High-point relaxation: We relax the lower-level optimality condition (i.e., drop the follower's objective), but ensure feasibility on the constraint region

$$\{(x, y) : A_1x + B_1y \geq b_1, A_2x + B_2y \geq b_2\}.$$

Properties of the high-point relaxation and the inducible region (IR) for linear bilevel programs (Bialas & Karwan, 1982; Bard, 1983):

- IR is a **union of faces** of the high-point relaxation,
- IR is a **connected set**, if there are **no coupling constraints**,
- if the program is feasible, then there exists an **optimal solution**, which is a **vertex of the high-point relaxation**.

Interval Linear Bilevel Programming: The First Attempt [Calvete & Galé, 2012]

Interval Linear Bilevel Programming

$$\begin{aligned} \min_{x,y} \quad & [c]^T x + [d]^T y \\ \text{s. t.} \quad & [A_1]x + [B_1]y \geq [b_1] \\ & y \in M(x) \end{aligned}$$

$$\text{where} \quad M(x) = \begin{aligned} \operatorname{argmin}_y \quad & [a]^T y \\ \text{s. t.} \quad & [A_2]x + [B_2]y \geq [b_2] \end{aligned}$$

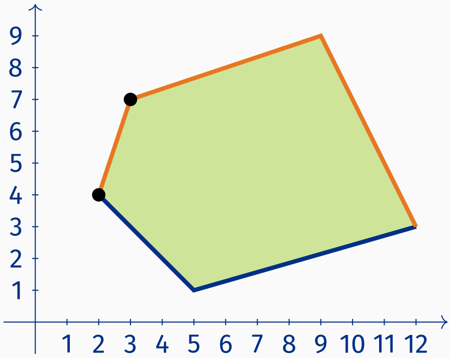
→ Compute the **best** and the **worst optimal value**, set of optimal solutions, etc., over all scenarios (bilevel programs) with coefficients from the given intervals.

Interval linear BP: Example I

$$\min_{x,y}\ [1,2]x + [-2,-1]y$$

$$y \in \arg\min_{y'} \{ [-1,2]y' :$$

$$(x,y') \in \operatorname{conv}\{(2,4),(3,7),(9,9),(12,3),(5,1)\}\}$$

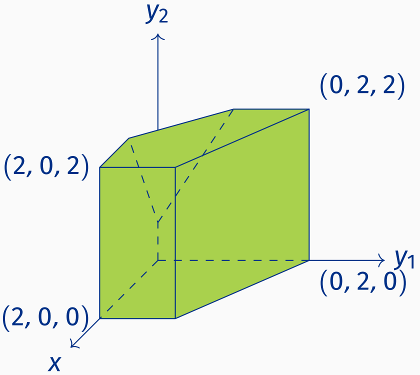


Interval linear BP: Example II

$$\min_{x,y}\; x + y_1 + y_2$$

$$y \in \arg\min_{y} \{[\underline{a_1}, \overline{a_1}]y_1 + [\underline{a_2}, \overline{a_2}]y_2 :$$

$$-3x - 3y_1 + 2y_2 \leq 1,\; x + 2y_1 \leq 4,\; y_2 \leq 2,\; x \leq 2,\; x, y_1, y_2 \geq 0\}$$



Bilevel Programming with Interval Objectives

$$\min_{x,y} \; [c]^T x + [d]^T y$$
$$y \in M(x)$$

$$\text{where } \; M(x) = \operatorname{argmin}_y \; [a]^T y$$
$$\text{s. t.} \qquad Ax + By \geq b, \;\; x, y \geq 0$$

Remark: The best and the worst optimal values of the bilevel interval problem occur at vertices of the constraint region

$$S = \{(x, y) : Ax + By \geq b, \;\; x, y \geq 0\}.$$

Upper-level Interval Objective Function

$$\min_{x,y} \quad [c]^T x + [d]^T y$$

$$y \in \arg\min_{y'} \{ a^T y' : Ax + By' \geq b, \; x, y \geq 0 \}$$

$\longrightarrow$ The **inducible region remains unchanged** for all scenarios.

Theorem

The best optimal value of the interval linear BP can be found by solving the scenario $(\underline{c}, \underline{d}, a)$.

The worst optimal value of the interval linear BP can be found by solving the scenario $(\overline{c}, \overline{d}, a)$.

Upper-level Interval Objective Function (cont'd)

Theorem

The best optimal value of the interval linear BP can be found by solving the scenario $(\underline{c}, \underline{d}, a)$.

Proof:

Let (x^l, y^l) be an optimal solution of the scenario $(\underline{c}, \underline{d}, a)$ and let (x^*, y^*) be optimal for some scenario (c, d, a) with $c \in [c]$, $d \in [d]$. Since the inducible region does not change, we have

$$\underline{c}^T x^l + \underline{d}^T y^l \leq \underline{c}^T x^* + \underline{d}^T y^* .$$

Moreover, since x, y are non-negative, we have

$$\underline{c}^T x^* + \underline{d}^T y^* \leq c^T x^* + d^T y^* .$$

Thus, the scenario $(\underline{c}, \underline{d}, a)$ yields the best optimal value. □

Lower-level Interval Objective Function

$$\min_{x,y} \; c^T x + d^T y$$
$$y \in \arg\min_{y'} \{ [a]^T y' : Ax + By' \geq b, \; x, y \geq 0 \}$$

Theorem

If the inducible region does not change for any vertex point of $[a]$, then the best and the worst optimal values coincide and can be found by solving the linear bilevel program

$$\min_{x,y} \; c^T x + d^T y$$
$$y \in \arg\min_{y'} \{ \underline{a}^T y' : Ax + By' \geq b, \; x, y \geq 0 \}.$$

KBB: Finding the Best Optimum

Consider the linear program on the **constraint region**

$$\min_{x,y} \; c^T x + d^T y : \; Ax + By \geq b, \; x, y \geq 0. \tag{1}$$

Let (x^*, y^*) be an optimal solution of (1). Then, (x^*, y^*) is the **best optimum** of the interval bilevel program, if it **belongs to the inducible region** for some $a^* \in [a]$, or, if the following linear system has a solution:

$$\begin{aligned} & u^T B \leq a, \quad a^T y^* = (b - Ax^*)^T u, \\ & \underline{a} \leq a \leq \overline{a}, \\ & u \geq 0. \end{aligned} \tag{2}$$

KBB: Finding the Best Optimum (cont'd)

In the following algorithm,

$W^{[i]}$: denotes the set of adjacent extreme points of $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ so that $(\mathbf{x}, \mathbf{y}) \in W^{[i]}$ implies $\mathbf{cx} + \mathbf{dy} \geqslant \mathbf{cx}^{[i]} + \mathbf{dy}^{[i]}$,
T: denotes the set of analyzed and discarded extreme points and
W: denotes the set of extreme points that will be analyzed later.

KBB algorithm:

Step1. Put $i \leftarrow 1$. Solve the linear programming problem (1) to obtain the optimal solution $(\mathbf{x}^{[1]}, \mathbf{y}^{[1]})$. Let $W \leftarrow \{(\mathbf{x}^{[1]}, \mathbf{y}^{[1]})\}$ and $T \leftarrow \emptyset$. Go to Step 2.
Step2. Set $(\mathbf{x}^*, \mathbf{y}^*) \leftarrow (\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ and check system (2). If the system is feasible, stop; $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ is the best optimal solution. Otherwise, go to Step 3.
Step3. Let $T \leftarrow T \cup \{(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})\}$ and $W \leftarrow (W \cup W^{[i]}) - T$.
Go to Step 4.
Step4. Set $i \leftarrow i + 1$ and choose $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ so that $\mathbf{cx}^{[i]} + \mathbf{dy}^{[i]} \leftarrow \min\{\mathbf{cx} + \mathbf{dy} : (\mathbf{x}, \mathbf{y}) \in W\}$. Go to Step 2.

"The ordered **best extreme points** of the constraint region
with respect to the upper-level objective
are sequentially computed (by **examining neighbors**),
until a **point of the inducible region** for some $a \in [a]$ is found."

KBB: Correctness of the method

Theorem

Let $(x^*, y^*) = (x^k, y^k)$ be the best extreme point of the constraint region S (with respect to the upper-level objective), so that there exists $a^* \in [a]$ such that $(x^*, y^*) \in IR_{a^*}$. Then, the point (x^*, y^*) is the best optimal solution of the interval linear BP.

Theorem

The algorithm KBB finds the best optimal solution of the interval linear bilevel program.

KBW: Finding the Worst Optimum (...or not?)

Again, we can use system (2) to check **feasibility** (whether (x^*, y^*) belongs to the inducible region for some $a \in [a]$):

$$u^T B \leq a, \quad a^T y^* = (b - Ax^*)^T u, \; \underline{a} \leq a \leq \overline{a}, \; u \geq 0.$$

To check **optimality** (i.e., if (x^*, y^*) is the solution for some $a \in [a]$), we consider the set

$$\Psi_{(x^*, y^*)} = \{a \in [a] : uG = a, u \geq 0\},$$

where $Gy = g$ is the subset of constraints $By \geq b - Ax$, $y \geq 0$, which are binding at (x^*, y^*).

KBW: Finding the Worst Optimum (cont'd)

KBW algorithm:

Step1. Put $i \leftarrow 1$. Solve the linear programming problem (9) to obtain the optimal solution $(\mathbf{x}^{[1]}, \mathbf{y}^{[1]})$.

$$\begin{aligned} &\max\ \mathbf{cx} + \mathbf{dy} \\ &s.t. \quad (\mathbf{x}, \mathbf{y}) \in S. \end{aligned} \tag{9}$$

Let $W \leftarrow \{(\mathbf{x}^{[1]},\ \mathbf{y}^{[1]})\}, W^e \leftarrow \emptyset, W^p \leftarrow \emptyset$ and $T \leftarrow \emptyset$. Go to Step 2.

Step2. Calculate $\psi_{(\mathbf{x}^{[i]},\mathbf{y}^{[i]})}$. If $\psi_{(\mathbf{x}^{[i]},\mathbf{y}^{[i]})} = \emptyset$, set $W^e \leftarrow W^e \cup \{(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})\}$. Go to Step 8. Otherwise, set $W^p \leftarrow W^{[i]} - (T \cup W^e)$. If $W^p = \emptyset$, stop; $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ is the worst optimal solution. Otherwise, set $\psi_* \leftarrow \psi_{(\mathbf{x}^{[i]},\mathbf{y}^{[i]})}$, go to Step 3.

Step3. If $W^p = \emptyset$, go to Step 8. Otherwise go to Step 4.

Step4. Choose $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ so that

$$\mathbf{c}\hat{\mathbf{x}} + \mathbf{d}\hat{\mathbf{y}} \leftarrow \max\{\mathbf{cx} + \mathbf{dy} : (\mathbf{x}, \mathbf{y}) \in W^p\}.$$

Calculate $\psi_{(\hat{\mathbf{x}},\hat{\mathbf{y}})}$. If $\psi_{(\hat{\mathbf{x}},\hat{\mathbf{y}})} = \emptyset$, set $W^e \leftarrow W^e \cup \{(\hat{x}, \hat{y})\},\ W^p \leftarrow W^p - \{(\hat{x}, \hat{y})\}$. Go to Step 3.

Step5. If $\psi^* \cap \psi_{(\hat{\mathbf{x}},\hat{\mathbf{y}})} = \emptyset$, set $W^p \leftarrow W^p - \{(\hat{x}, \hat{y})\}$, go to Step 3. Otherwise, go to Step 6.

Step6. Set $\psi^* \leftarrow \psi^* - \psi_{(\hat{\mathbf{x}},\hat{\mathbf{y}})}$. If $\psi^* = \emptyset$, set $T \leftarrow T \cup \{(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})\}$, go to Step 8. Otherwise, go to Step7.

Step7. Set $W^p \leftarrow W^p - \{(\hat{\mathbf{x}}, \hat{\mathbf{y}})\}$. If $W^p = \emptyset$, stop; $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ is the worst optimal solution. Otherwise, go to Step 4.

Step8. Set $W \leftarrow (W \cup W^{[i]}) - (T \cup W^e)$. Set $i \leftarrow i + 1$ and choose $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ so that $\mathbf{cx}^{[i]} + \mathbf{dy}^{[i]} \leftarrow \max\{\mathbf{cx} + \mathbf{dy} : (\mathbf{x}, \mathbf{y}) \in W\}$. Go to Step 2.

Similar idea to KBB, but now both **feasibility and optimality** conditions for the solution have to be tested in the algorithm (only feasibility is not sufficient).

Upper- and Lower-level Interval Objectives

$$\min_{x,y} \; [c]^T x + [d]^T y$$
$$y \in \arg\min_{y'} \{ [a]^T y' : Ax + By' \geq b, \; x, y \geq 0 \}$$

Theorem

*The **best optimal solution** of the interval bilevel program is obtained by applying the KBB algorithm to the program with upper-level objective vectors $c = \underline{c}$, $d = \underline{d}$.*

*The **worst optimal solution** of the interval bilevel program is obtained by applying the KBW algorithm to the problem with upper-level objective vectors $c = \overline{c}$, $d = \overline{d}$.*

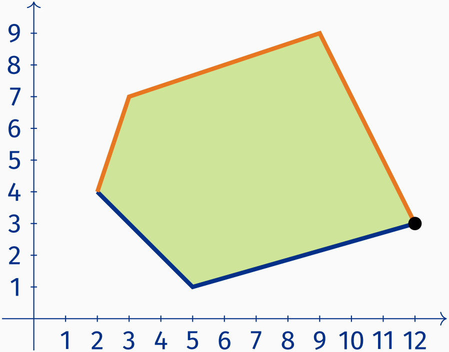
Interval Linear Bilevel Programming: The Revised Algorithm [Mishmast Nehi & Hamidi, 2015]

Counterexample I

$$\min_{x,y} \; 0x + 1y$$

$$y \in \arg\min_{y'} \; \{[-1, 2]y' :$$

$$(x, y') \in \text{conv}\{(2,4), (3,7), (9,9), (12,3), (5,1)\}\}$$

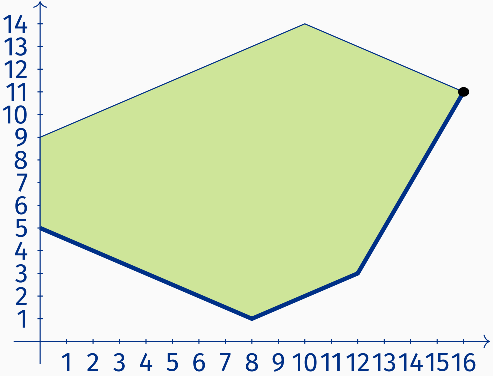


Counterexample II

$$\min_{x,y} \; 0x - y$$

$$y \in \arg\min_{y'} \{ [1,2]y' :$$

$$(x, y') \in \operatorname{conv}\{ (0,9), (10,14), (16,11), (12,3), (8,1), (0,5) \} \}$$



Revised KBW

value of the upper level objective function with this process. The general process of KBW algorithm is correct; in this algorithm use is made of the W^p to study candidate point $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$, but what is to be noted in this algorithm is that to study point $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$, not only its adjacent extreme points, but also all those extreme points (remained from W^p in the previous iterations) that have not been analyzed or have not been discarded should be evaluated too. The step 2 shortcoming in KBW algorithm in relation (16) has been corrected in the proposed revised algorithm. On the one hand, point $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ cannot be used in its own evaluation; so, in order to solve these problems, it is necessary that relation

$$W^p \leftarrow W^{[i]} - (T \cup W^e) \tag{15}$$

in KBW algorithm be modified as follows:

$$W^p \leftarrow W^p \cup W^{[i]} - (T \cup W^e \cup \{(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})\}) \tag{16}$$

And, on the other hand, no precise definition is given for $W^{[i]}$ in the paper. We can have the following more precise definition:

$W^{[i]}$: denotes the set of adjacent extreme points of $(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})$ so that for each $(\mathbf{x}, \mathbf{y}) \in W^{[i]}$:

$$\begin{cases} \mathbf{cx} + \mathbf{dy} \leqslant \mathbf{cx}^{[i]} + \mathbf{dy}^{[i]}, & \text{if } \psi_{(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})} = \emptyset. \\ \mathbf{cx} + \mathbf{dy} < \mathbf{cx}^{[i]} + \mathbf{dy}^{[i]}, & \text{if } \psi_{(\mathbf{x}^{[i]}, \mathbf{y}^{[i]})} \neq \emptyset. \end{cases} \tag{17}$$

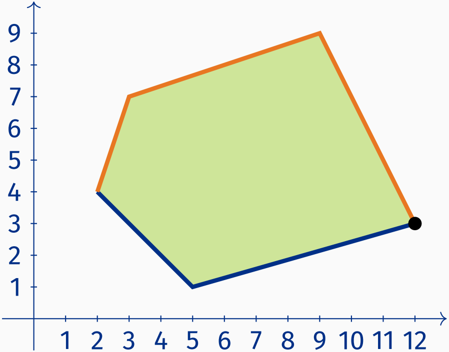
H. Mishmast Nehi, F. Hamidi / Applied Mathematical Modelling 39 (2015) 1650–1664 1657

Revised KBW: Example

$$\min_{x,y} \; 0x + 1y$$

$$y \in \arg\min_{y'} \; \{ [-1, 2] y' :$$

$$(x, y') \in \operatorname{conv}\{ (2,4), (3,7), (9,9), (12,3), (5,1) \} \}$$



Fully Interval Linear Bilevel Programming

$$
\begin{aligned}
\min_{x,y} \quad & [c]^T x + [d]^T y \\
\text{s. t.} \quad & [A_1]x + [B_1]y \geq [b_1] \\
& x, y \geq 0 \\
& y \in M(x)
\end{aligned}
$$

$$
\begin{aligned}
\text{where } M(x) = \operatorname{argmin}_y \quad & [a]^T y \\
\text{s. t.} \quad & [A_2]x + [B_2]y \geq [b_2] \\
& y \geq 0
\end{aligned}
$$

Fully Interval BLP: the Best and the Worst

Consider the constraint region

$$\{(x, y) : [A_1]x + [B_1]y \geq [b_1],\ [A_2]x + [B_2]y \geq [b_2],\ x, y \geq 0\}.$$

Remark: The best and the worst optimal value of the fully interval bilevel linear program occur at vertices of $\overline{S}$, $\underline{S}$, respectively, where

$$\overline{S} = \{(x, y) : \overline{A_1}x + \overline{B_1}y \geq \underline{b_1},\ \overline{A_2}x + \overline{B_2}y \geq \underline{b_2},\ x, y \geq 0\},$$
$$\underline{S} = \{(x, y) : \underline{A_1}x + \underline{B_1}y \geq \overline{b_1},\ \underline{A_2}x + \underline{B_2}y \geq \overline{b_2},\ x, y \geq 0\}.$$

Fully Interval BLP: the Best and the Worst (cont'd)

Theorem

*The **best** and the **worst optimal values** of the fully interval bilevel linear program are the optimal values of the following problems, respectively:*

$$
\begin{aligned}
\min_{x,y}\;\; & \underline{c}^T x + \underline{d}^T y \\
\text{s. t.}\;\; & \overline{A_1} x + \overline{B_1} y \geq \underline{b_1}, \\
& x, y \geq 0 \\
& y \in \arg\min \{\ [a]^T y : \\
& \qquad \overline{A_2} x + \overline{B_2} y \geq \underline{b_2}, \\
& \qquad y \geq 0\ \}
\end{aligned}
\qquad\qquad
\begin{aligned}
\min_{x,y}\;\; & \overline{c}^T x + \overline{d}^T y \\
\text{s. t.}\;\; & \underline{A_1} x + \underline{B_1} y \geq \overline{b_1}, \\
& x, y \geq 0 \\
& y \in \arg\min \{\ [a]^T y : \\
& \qquad \underline{A_2} x + \underline{B_2} y \geq \overline{b_2}, \\
& \qquad y \geq 0\ \}
\end{aligned}
$$

→ Extended KBB and RKBW algorithms

Conclusion

- Methods for computing **the best and the worst optimal values** for bilevel linear programs with interval objectives are available.

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- Still, many related questions and problems in interval bilevel programming remain open.

Conclusion

- Methods for computing **the best and the worst optimal values** for bilevel linear programs with interval objectives are available.
- These methods were also further extended to be used for fully interval bilevel linear programs.
- Still, many related questions and problems in interval bilevel programming remain open.

Thank you for your attention!

References

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