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Completely positive matrices

Quadratic forms:

- $Q\colon \mathbb{R}^n \to \mathbb{R}$
- $Q(x_1, \ldots, x_n) := \mathbf{x}^T Q \mathbf{x} = \sum_{i,j} q_{ij} x_i x_j$
  - $Q \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$
- Special classes of forms:
  1. positive semi-definite forms (PSD)
     - $\forall \mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T Q \mathbf{x} \geq 0$
  2. forms with non-negative coefficient (NNC)
     - $\mathbf{x}^T N \mathbf{x} = \sum_{i,j} n_{ij} x_i x_j$
     - $\forall i, j : n_{ij} \geq 0$

Completely positive forms

$Q(\mathbf{x}) := \mathbf{x}^T A \mathbf{x}$ is ***completely positive***, if

1. $\exists B \in \mathbb{R}^{n \times m} : A = BB^T$,
2. $B \geq 0_{n \times m}$.

- $B \geq 0_{n \times m} \implies A = BB^T \geq 0_{n \times n}$
- $\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T BB^T \mathbf{x} = (B^T \mathbf{x})^T (B^T \mathbf{x}) = \sum_{i=1}^{m} (B^T \mathbf{x})_i^2 \geq 0$

Completely positive forms are

1. (PSD) positive semi-definite forms
2. (NNC) forms with non-negative coefficients

Alternative characterisation of CP forms

- $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T BB^T\mathbf{x} = (B^T\mathbf{x})^T(B^T\mathbf{x}) = \sum_{i=1}^{m}(B^T\mathbf{x})_i^2$
- i-th coordinate of $(B^T\mathbf{x})$:
  - $(B^T\mathbf{x})_i = \sum_{k=1}^{n} B_{ik}^T x_k = \sum_{k=1}^{n} B_{ki}x_k$
  - $L_i(\mathbf{x}) := \sum_{k=1}^{n} B_{ki}x_k$ ... form with non-negative coefficients
    - *linear* form
- $Q(\mathbf{x}) = \sum_{i=1}^{m} L_i^2(\mathbf{x})$

Alternative characterisatino of CP forms

- $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T B B^T \mathbf{x} = (B^T \mathbf{x})^T (B^T \mathbf{x}) = \sum_{i=1}^{m} (B^T \mathbf{x})_i^2$
- i-th coordinate of $(B^T \mathbf{x})$:
  - $(B^T \mathbf{x})_i = \sum_{k=1}^{n} B^T_{ik} x_k = \sum_{k=1}^{n} B_{ki} x_k$
  - $L_i(\mathbf{x}) := \sum_{k=1}^{n} B_{ki} x_k$ ... form with non-negative coefficients
    - *linear* form
- $Q(\mathbf{x}) = \sum_{i=1}^{m} L_i^2(\mathbf{x})$

Characterisation of CP forms

A form $Q(\mathbf{x})$ is completely positive if and only if there is $m \in \mathbb{N}$:

$$Q(\mathbf{x}) = \sum_{i=1}^{m} L_i^2(\mathbf{x}),$$

where $L_i(\mathbf{x})$ for $i = 1, \ldots, m$ are non-negative forms.

Matrix represetation of $L_i(\mathbf{x})$

- $Q(\mathbf{x}) = \sum_{i=1}^{m} L_i^2(\mathbf{x})$
- $L_i^2(\mathbf{x}) = \left(\sum_{j=1}^{n} \ell_j x_j\right)^2 = \sum_{i,j=1}^{n} \ell_i \ell_j x_i x_j = \mathbf{x}^T L_i \mathbf{x}$
  - $L_i = \boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^T$
  - $(\boldsymbol{\lambda}_i)_j := \ell_j$
- $A = L_1 + \cdots + L_m = \sum_{i=1}^{m} \boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^T$
- $Q(\mathbf{x}) = \sum_{i=1}^{m} \mathbf{x}^T \boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^T \mathbf{x} = \sum_{i=1}^{m} (\lambda_i^T \mathbf{x})(\boldsymbol{\lambda}_i^T \mathbf{x}) = \sum_{i=1}^{m} (\lambda_i^T \mathbf{x})^2$

Characterisation of CP forms

A form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is completely positive if and only if there is $m \in \mathbb{N}$:

- $Q(\mathbf{x}) = \sum_{i=1}^{m} L_i^2(\mathbf{x})$
- $Q(\mathbf{x}) = \sum_{i=1}^{m} \mathbf{x}^T \boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^T \mathbf{x}$

where $L_i(\mathbf{x})$ for $i = 1, \dots, m$ are non-negative forms and $\boldsymbol{\lambda}_i \boldsymbol{\lambda}_i$ their matrix representations.

- How big is m?
- $\implies$ *cp-rank* of A:
- minimal $m \in \mathbb{N}$ such that
  1. $\exists L_1(\mathbf{x}), \dots, L_m(\mathbf{x}) : Q(\mathbf{x}) = \sum_{i=1}^{m} L_i^2(\mathbf{x})$
  2. $\exists \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m : A = \sum_{i,j=1}^{m} \boldsymbol{\lambda}_i \boldsymbol{\lambda}_i^T$
  3. $\exists B \in \mathbb{R}^{n \times m} : A = BB^T$

- $CP^n = \{Q(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R} \mid Q(\mathbf{x}) \text{ if completely positive}\}$

Properties of CP^n:

1. $Q_1(\mathbf{x}), Q_2(\mathbf{x}) \in CP^n \implies Q_3(\mathbf{x}) := Q_1(\mathbf{x}) + Q_2(\mathbf{x}) \in CP^n$
   - $Q_1(\mathbf{x}) = \sum_{i=1}^{m_1} L_i^2(\mathbf{x})$
   - $Q_2(\mathbf{x}) = \sum_{i=1}^{m_2} M_i^2(\mathbf{x})$
   - $Q_3(\mathbf{x}) = Q_1(\mathbf{x}) + Q_2(\mathbf{x}) = \sum_{i=1}^{m_1} L_i^2(\mathbf{x}) + \sum_{j=1}^{m_2} M_j^2(\mathbf{x})$
     - $Q_1(\mathbf{x}) \sim A = BB^T$, $Q_2(\mathbf{x}) \sim C = DD^T$
     - $L_i(\mathbf{x}) \sim B_{*i}$, $M_j(\mathbf{x}) \sim C_{*j}$
   - $E := \begin{pmatrix} B & D \end{pmatrix}$
   - $EE^T = \begin{pmatrix} B & D \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} = BB^T + DD^T = A + C$
   - $Q_3(\mathbf{x}) = \mathbf{x}^T EE^T \mathbf{x}$
   - $Q_3(\mathbf{x}) = \mathbf{x}^T (A + C) \mathbf{x}$

- $CP^n = \{Q(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R} \mid Q(\mathbf{x}) \text{ if completely positive}\}$

Properties of CP^n:

1. $Q_1(\mathbf{x}), Q_2(\mathbf{x}) \in CP^n \implies Q_3(\mathbf{x}) := Q_1(\mathbf{x}) + Q_2(\mathbf{x}) \in CP^n$
2. $Q_1(\mathbf{x}) \in CP^n, a \geq 0 \implies Q_2(\mathbf{x}) := aQ_1(\mathbf{x}) \in CP^n$
    - $Q_1(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T BB^T\mathbf{x}$
    - $Q_2(\mathbf{x}) = \mathbf{x}^T aA\mathbf{x} = \mathbf{x}^T aBB^T\mathbf{x} = \mathbf{x}^T(\sqrt{a}B)(B^T\sqrt{a})\mathbf{x}$
        - $b := \sqrt{a}$
        - $B_2 := \sqrt{a}B$

CP^n is a convex cone

CP^n forms a closed convex cone.

Standard quadratic programming:

$$\begin{aligned} \min \quad & \mathbf{x}^T Q \mathbf{x} \\ s.t. \quad & \mathbf{e}^T \mathbf{x} = 1, \\ & \mathbf{x} \geq 0. \end{aligned}$$

- NP-hard optimization problem
  - reduction of max-clique graph problem
- $\mathbf{x}^T Q \mathbf{x} = Q\mathbf{x}\mathbf{x}^T = tr(Q^T \mathbf{x}\mathbf{x}^T) = \langle Q, \mathbf{x}\mathbf{x}^T \rangle_F$
  - $\min \mathbf{x}^T Q \mathbf{x} \rightarrow \min \langle Q, \mathbf{x}\mathbf{x}^T \rangle_F$
- $\mathbf{e}^T \mathbf{x} = 1 \rightarrow \langle \mathbf{e}\mathbf{e}^T, \mathbf{x}\mathbf{x}^T \rangle_F = 1$
  - $\langle \mathbf{e}\mathbf{e}^T, \mathbf{x}\mathbf{x}^T \rangle_F = tr\left((\mathbf{e}\mathbf{e}^T)^T (\mathbf{x}\mathbf{x}^T) \right) = tr\left(\mathbf{e}(\mathbf{e}^T\mathbf{x})\mathbf{x}^T \right) =$
  - $= \sum_i (\mathbf{e}\mathbf{x}^T)_{ii} = \sum_i \mathbf{e}_i \mathbf{x}_i = \mathbf{e}^T \mathbf{x} = 1$

Special CP programming:

$$\begin{aligned} \min \quad & \langle Q, \mathbf{x}\mathbf{x}^T \rangle_F \\ s.t. \quad & \langle \mathbf{e}\mathbf{e}^T, \mathbf{x}\mathbf{x}^T \rangle_F = 1, \\ & \mathbf{x} \geq 0. \end{aligned}$$

- NP-hard optimization problem
  - reduction of max-clique graph problem
- $\mathbf{x}^T Q \mathbf{x} = Q\mathbf{x}\mathbf{x}^T = tr(Q^T \mathbf{x}\mathbf{x}^T) = \langle Q, \mathbf{x}\mathbf{x}^T \rangle_F$
  - $\min \mathbf{x}^T Q \mathbf{x} \to \min \langle Q, \mathbf{x}\mathbf{x}^T \rangle_F$
- $\mathbf{e}^T \mathbf{x} = 1 \to \langle \mathbf{e}\mathbf{e}^T, \mathbf{x}\mathbf{x}^T \rangle_F = 1$
  - $\langle \mathbf{e}\mathbf{e}^T, \mathbf{x}\mathbf{x}^T \rangle_F = tr\left((\mathbf{e}\mathbf{e}^T)^T (\mathbf{x}\mathbf{x}^T)\right) = tr\left(\mathbf{e}(\mathbf{e}^T\mathbf{x})\mathbf{x}^T\right) =$
  - $= \sum_i (\mathbf{e}\mathbf{x}^T)_{ii} = \sum_i \mathbf{e}_i \mathbf{x}_i = \mathbf{e}^T \mathbf{x} = 1$

Special CP programming:

$$\begin{aligned} \min \quad & \langle Q, X \rangle_F \\ s.t. \quad & \langle \mathbf{ee}^T, X \rangle_F = 1, \\ & {\color{red} X = \mathbf{xx}^T} \\ & \mathbf{x} \geq 0 \end{aligned}$$

- NP-hard optimization problem
  - reduction of max-clique graph problem
- $\mathbf{x}^T Q \mathbf{x} = Q\mathbf{xx}^T = tr(Q^T \mathbf{xx}^T) = \langle Q, \mathbf{xx}^T \rangle_F$
  - $\min \mathbf{x}^T Q \mathbf{x} \rightarrow \min \langle Q, \mathbf{xx}^T \rangle_F$
- $\mathbf{e}^T \mathbf{x} = 1 \rightarrow \langle \mathbf{ee}^T, \mathbf{xx}^T \rangle_F = 1$
  - $\langle \mathbf{ee}^T, \mathbf{xx}^T \rangle_F = tr\left((\mathbf{ee}^T)^T (\mathbf{xx}^T)\right) = tr\left(\mathbf{e}(\mathbf{e}^T \mathbf{x})\mathbf{x}^T\right) =$
  - $= \sum_i (\mathbf{ex}^T)_{ii} = \sum_i \mathbf{e}_i \mathbf{x}_i = \mathbf{e}^T \mathbf{x} = 1$

CP relaxation:

$$\begin{aligned} \min \quad & \langle Q, X \rangle_F \\ s.t. \quad & \langle \mathbf{e}\mathbf{e}^T, X \rangle_F = 1, \\ & X \in CP^n \end{aligned}$$

- NP-hard optimization problem
  - reduction of max-clique graph problem
- actually $\mathbf{x}\mathbf{x}^T, \mathbf{x} \geq 0 \rightarrow X \in CP^n$ is not a relaxation
  - $CP^n = conv\{\mathbf{x}\mathbf{x}^T \mid \mathbf{x} \in \mathbb{R}^n_+\}$
  - linear cost function + convex set $\implies$ optimum attained in a vertex

Completely positive programming

Completely positive programming:

$$
\begin{aligned}
\min \qquad & \langle C, X\rangle_F \\
s.t. \qquad & \langle A_i, X\rangle_F = \mathbf{b}_i, \\
& X \in CP^n, \\
& i = 1, \ldots, m
\end{aligned}
$$

Lagrangian dual:

$$
\begin{aligned}
\max \qquad & \sum_{i=1}^{m} b_i y_i \\
s.t. \qquad & C - \sum_{i=1}^{m} \langle A_i, X\rangle_F = \underline{i}, \\
& X \in CP^n_*
\end{aligned}
$$

- CP^n_* ... dual cone

- $\langle A, B\rangle_F := tr(A^T B) = \sum_{i,j=1}^{n} a_{ij} b_{ij}$
- $CP_*^n = \{A \in \mathbb{R}^{n\times n} \mid \langle A, B\rangle_F \geq 0, \forall B \in CP^n\}$
  - $B = \mathbf{x}\mathbf{x}^T$
    - $\mathbf{x} \in \mathbb{R}_+^n$
    - $b_{ij} = x_i x_j$
  - $0 \leq \langle A, B\rangle_F = \sum_{i,j=1}^{n} a_{ij} b_{ij} = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^T A \mathbf{x}$

Frobenius inner product and dual cone

- $\langle A, B\rangle_F := tr(A^T B) = \sum_{i,j=1}^{n} a_{ij}b_{ij}$
- $CP_*^n = \{A \in \mathbb{R}^{n\times n} \mid \langle A, B\rangle_F \geq 0, \forall B \in CP^n\}$
  - $B = \mathbf{x}\mathbf{x}^T$
    - $\mathbf{x} \in \mathbb{R}_+^n$
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  - $0 \leq \langle A, B\rangle_F = \sum_{i,j=1}^{n} a_{ij}b_{ij} = \sum_{i,j=1}^{n} a_{ij}x_i x_j = \mathbf{x}^T A\mathbf{x}$

Q($\mathbf{x}$)=$\mathbf{x}^T A\mathbf{x}$ is ***copositive*** (constrained positive), if

$$\forall \mathbf{x} \in \mathbb{R}_+^n : \mathbf{x}^T A\mathbf{x} \geq 0$$

- $A \in CP_*^n \implies A$ is copositive
- Copositive forms form a closed convex cone:
  - $\mathbf{x}^T(A + B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} \geq 0$
  - $b \geq 0 : \mathbf{x}^T(bA)\mathbf{x} = b\mathbf{x}^T A\mathbf{x} \geq 0$

Frobenius inner product and dual cone

- A is copositive $\overset{?}{\implies}$ $A \in CP_*^n$
  - $\langle A, \mathbf{x}\mathbf{x}^T \rangle \geq 0$ for $\mathbf{x} \in \mathbb{R}_+^n$
  - $B = \sum_{i=1}^m \mathbf{x}_i \mathbf{x}_i^T$
  - $\langle A, B \rangle = \langle A, \sum_{i=1}^m \mathbf{x}_i \mathbf{x}_i^T \rangle = \sum_{i=1}^n \langle A, \mathbf{x}_i \mathbf{x}_i^T \rangle \geq 0$

Frobenius inner product and dual cone

- A is copositive $\overset{?}{\Longrightarrow}$ $A \in CP_*^n$
  - $\langle A, \mathbf{x}\mathbf{x}^T \rangle \geq 0$ for $\mathbf{x} \in \mathbb{R}_+^n$
  - $B = \sum_{i=1}^m \mathbf{x}_i \mathbf{x}_i^T$
  - $\langle A, B \rangle = \langle A, \sum_{i=1}^m \mathbf{x}_i \mathbf{x}_i^T \rangle = \sum_{i=1}^n \langle A, \mathbf{x}_i \mathbf{x}_i^T \rangle \geq 0$

Duality of cones of CP and COP

The convex cones of completely positive forms and copositive forms are duals under Frobenius inner product.

- still not properly answered
- sufficient and necessary conditions
    - through connection of CP to graph theory

- $G = (V, E)$ ... graph
  - V ... vertices
  - E ... edges
- $B \in \mathbb{R}^{|V| \times |E|}$ ... incidence matrix
- $B_{ve} := \begin{cases} 1, & \text{if } v \in e, \\ 0, & \text{if } v \notin e. \end{cases}$
- $v, w \in V : (BB^T)_{vw} = \langle b_{v*}, b_{w*} \rangle$
  - $(BB^T)_{vw} = \sum_{e=1}^{|E|} B_{ve} B^T_{ew} = \sum_{e=1}^{|E|} b_{ve} b_{we}$
- $\langle b_{v*}, b_{w*} \rangle = \begin{cases} 1, & \text{if } v \neq w, \{v, w\} \in E, \\ 0, & \text{if } v \neq w, \{v, w\} \notin E, \\ \deg(v), & \text{if } v = w. \end{cases}$
- $A - diag(A)$ ... adjacency matrix of graph G

- A is **diagonally dominant** if $\forall i : a_{ii} > \sum_{i \neq j} a_{ij}$

Sufficient condition of CP

Let A be a non-negative diagonally dominant symmetric matrix. Then $A \in CP^n$.

- Idea: Assign G to A
  - incidence matrix of $G \implies$ matrix B s.t. $A = BB^T$
- $G(A) := (V, E)$
  - $V = \{1, \ldots, n\}$
  - $E = \{\{i, j\} \mid a_{ij} > 0\}$
    - actually a multi-graph ($i = j$)
- B ... incidence matrix of $G(A)$
  - $b_{ve} := 1$, if $v \in e$
  - $i \neq j : (BB^T)_{ij} = 1 \iff a_{ij} > 0$
  - $i = j : (BB^T)_{ii} = deg(i) + 1$

A SUFFICIENT CONDITION OF CP

- A is **diagonally dominant** if $\forall i : a_{ii} > \sum_{i \neq j} a_{ij}$

Sufficient condition of CP

Let A be a non-negative diagonally dominant symmetric matrix. Then $A \in CP^n$.

- Idea: Assign G to A
  - incidence matrix of $G \implies$ matrix B s.t. $A = BB^T$
- $G(A) \coloneqq (V, E)$
  - $V = \{1, \dots, n\}$
  - $E = \{\{i, j\} \mid a_{ij} > 0\}$
    - actually a multi-graph ($i = j$)
- B' ... modified incidence matrix of $G(A)$
  - $b'_{ve} \coloneqq \sqrt{a_{ij}}$, if $v \in e$, $|e| = 2$ and $b'_{ve} \coloneqq 1$, if $v \in e$, $|e| = 1$
  - $i \neq j : (B'B'^T)_{ij} = a_{ij} \iff a_{ij} > 0$
  - $i = j : (B'B'^T)_{ii} = \sum_{i \neq j} a_{ij} + 1$

- A is **diagonally dominant** if $\forall i : a_{ii} > \sum_{i \neq j} a_{ij}$

Sufficient condition of CP

Let A be a non-negative diagonally dominant symmetric matrix. Then $A \in CP^n$.

- Idea: Assign G to A
  - incidence matrix of $G \implies$ matrix B'' s.t. $A = B''B''^T$
- $G(A) := (V, E)$
  - $V = \{1, \dots, n\}$
  - $E = \{\{i, j\} \mid a_{ij} > 0\}$
    - actually a multi-graph ($i = j$)
- B'' ... modified' incidence matrix of $G(A)$
  - $b''_{ve} := 1$, if $v \in e$, $|e| = 2$ and $b''_{ve} := a_{ii} - \sum_{i \neq j} a_{ij}$, if $v \in e$, $|e| = 1$
  - $i \neq j : (B''B''^T)_{ij} = \sqrt{a_{ij}} \iff a_{ij} > 0$
  - $i = j : (B''B''^T)_{ii} = \sum_{i \neq j} a_{ij} + a_{ii} - \sum_{i \neq j} a_{ij} = a_{ii}$

- $G(A)$ … a graph **associated** to A
  - $G(A) := (V, E)$
    - $V = \{1, \ldots, n\}$
    - $E = \{\{i, j\} \mid a_{ij} > 0\}$
- $A = A^T$ … a **realisation** of G
  - A matrix A s.t. $G(A) = G$

- $G(A)$ ... a graph **associated** to A
  - $G(A) := (V, E)$
    - $V = \{1, \ldots, n\}$
    - $E = \{\{i, j\} \mid a_{ij} > 0\}$
- $A = A^T$ ... a **realisation** of G (or $G(A)$)
  - $\{i, j\} \in E \implies a_{ij} > 0$

Completely positive graphs

A graph G does not contain an odd cycle of length more than 3 if and only if every realisation A of G is completely positive.

Application:

- $G(A)$ does not contain the cycle $\implies A \in CP^n$

- $M(A)$ … a **comparison matrix** of A
  - $M(A)_{ij} := \begin{cases} a_{ij}, & \text{if } i = j, \\ -a_{ij}, & \text{if } i \neq j. \end{cases}$

- $M(A)$ ... a **comparison matrix** of A
  - $M(A)_{ij} := \begin{cases} a_{ij}, & \text{if } i = j, \\ -a_{ij}, & \text{if } i \neq j. \end{cases}$

$M(A)$ is PSD $\implies$ A is CP

For $A = A^T$, $M(A)$ is PSD $\implies$ A is CP.

Proof:

1. $M(A)$ is M-matrix $\implies$ A is CP
   - key element of the proof
2. $M(A)$ is PD $\implies$ $M(A)$ is M-matrix
   - Z-matrix is PD $\implies$ Z-matrix is M-matrix
3. $M(A)$ is PSD $\implies$ $M(A)$ is PD

$M(A)$ is PSD $\overset{?}{\Leftarrow}$ A is CP

- When does the opposite inequality hold?
  - we employ the graph $G(A)$

- When does the opposite inequality hold?
  - we employ the graph $G(A)$

$M(A)$ is PSD $\Leftarrow$ A is CP

For a graph G it holds for every matrix realisation A that

$$A \text{ is } CP \implies M(A) \text{ is } PSD$$

if and only if G is triangle free.

- Application:for $A \overset{?}{\in} CP^n$:
  1. construct G
  2. If G is triangle free:
     - $M(A)$ is PSD $\implies A \in CP^n$
     - $M(A)$ is not PSD $\implies A \notin CP^n$

Completely positive forms

- $Q(\mathbf{x}) = \sum_{i=1}^{n} L_i(\mathbf{x})$
  - $L_i(\mathbf{x})$ ... non-negative forms

Completely positive matrices

- $A = BB^T$
  - $B \in \mathbb{R}_{+}^{n \times m}$ ... non-negative matrices
- $A = \sum_{i=1} \mathbf{x}_i \mathbf{x}_i^T$
  - $\mathbf{x}_i \in R_{+}^{n}$ ... non-negative vectors

Completely positive forms

- $Q(\mathbf{x}) = \sum_{i=1}^{n} L_i(\mathbf{x})$

Completely positive matrices

- $A = BB^T$
- $A = \sum_{i=1} \mathbf{x}\mathbf{x}^T$

CP^n **... set of completely positive matrices**

- closed convex cone
- dual to the cone copositive matrices
  - studied in optimization
  - NP-hard optimization (both primal and dual)

Characterisation of CP

- open problem
- sufficient and necessary conditions
  - based on graphs $G(A)$
  - based on comparison matrices $M(A)$