Matchings and game theory

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Optimization seminar

Today's agenda

I will speak about **matchings**, i.e. about markets in which we assign some agents to other agents (moreover not arbitrarily but in a **stable** way with respect to their **preferences**). As this is quite abstract concept, we shall see **several possible models**.

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Market design is a practical methodology for creation of markets of certain properties, which is partially based on mechanism design. In some markets, prices may be used to induce the desired outcomes — these markets are the study of auction theory. In other markets, prices may not be used — these markets are the study of **matching theory**. *(source: Wikipedia)*

- Centralised vs. decentralised markets.
- Rules.
- Money does not solve everything.
- Ethics.
- Closed and specific markets.

Matching under preferences

Setting/model

We shall mainly speak about one-to-one matchings with preferences, the so-called **marriage markets**.

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- Players are **men**, $M = \{m_i, \ldots, m_n\}$, and **women**, $W = \{w_1, \ldots, w_n\}$.
- The market is two-sided, which means that m_i can have only preferences over the set W and analogously for women.
- Each man and woman has a **preference relation** on the respective opposite set. The preferences are strict (weakly antisymmetric), complete, reflexive and transitive relations.
- *(Some generalisations of these assumptions will be discussed later)*

Definition (Matching)

Given a marriage market with M and W, **matching** is a bijective mapping $\mu : M \cup W \rightarrow M \cup W$ such that:

- we have $\mu(M) \subseteq W$ and $\mu(W) \subseteq M$, and
- for every $m \in M$, $\mu(m) = w$, if and only if $\mu(w) = m$.

Definition (Stable matching)

We say that a matching μ is **stable** if there is no blocking pair (m, w), such that $m \in M$, $w \in W$ and simultaneously $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

If there is a blocking pair, the proposed matching would be simply blocked or refused by the blocking pair and the solution would fall apart.

Our main task is to **search for stable matchings**.

The deferred acceptance algorithm (DA)

- Step 1:
  - Each man $x \in M$ proposes to his first choice.
  - Each woman looks at who proposed. If she has more than one proposal, she "holds" the most preferred for her. (This is **deferred acceptance**.)
- Step k, $k \geq 2$:
  - Any man who has been rejected in the previous step makes a new proposal to his most preferred mate who has not yet rejected him.
  - Each woman holds again her most preferred offer and discards (rejects) the others.
- Stopping condition and procedure:
  - When no further proposals are made.
  - Match each woman to the man (if any) whose proposal she is holding.

The algorithm is also called **Gale-Shapley algorithm**.

- Gale, D., & Shapley, L. S. (1962). *College admissions and the stability of marriage*. The American Mathematical Monthly, 69(1), 9-15.

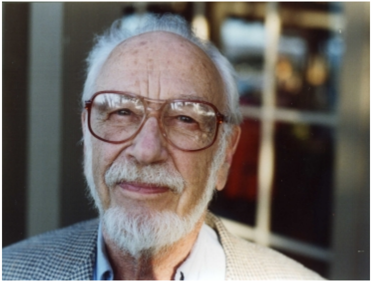

Figure: David Gale on the left, Lloyd Shapley on the right.

Theorem

For every marriage market, the Gale-Shapley algorithm finds a stable matching.

Proof (Part I).

The proof consists of three claims.

- *Claim 1: The GS algorithm is finite.*
  In each step after the first, every rejected man proposes to his next choice. No man tries to propose to a woman more than once. Since each man's preference relation is finite, the algorithm eventually stops.
- *Claim 2: The result is always a matching.*
  In the end, there cannot be a man and a woman both not engaged, as he must have proposed to her at some point (since a man will eventually propose to everyone, if necessary) and, being proposed to, she would necessarily be engaged (to someone) thereafter.

Proof (Part II).

- *Claim 3: If the result is matching, it is stable.*
  Let Alice and Bob both be engaged, but not to each other. Upon completion of the algorithm, it is not possible for both Alice and Bob to prefer each other over their current partners. If Bob prefers Alice to his current partner, he must have proposed to Alice before he proposed to his current partner. If Alice accepted his proposal, yet is not married to him at the end, she must have discard him for someone she likes more, and therefore doesn't like Bob more than her current partner. If Alice rejected his proposal, she was already with someone she liked more than Bob. □

Other aspects and generalisations

Complexity

The algorithm is clearly polynomial (to be precise it is quadratic) with respect to the size of input.

Generalisations

1. The agent can be indifferent between any two acceptable mates. I.e. its preference relation is not strict.
2. The agent can prefer to be unmatched than to be paired with certain other agents (*unacceptable matches*).

The GS algorithm still works for such generalisations (but one has to be careful with proofs and statements as there for example does not have to be always a stable matching).

Stable mechanism

Definition

We say that a function mapping marriage markets to matchings is a **stable mechanism** if it assigns to every situation a stable matching.

Truthfulness

What if someone is not truthful and lies?

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(We say that $w \in W$ is **achievable** for $m \in M$ if there exists a stable matching μ such that $\mu(m) = w$.)

Theorem

When any stable mechanism is applied to a marriage market in which preferences are strict and there is more than one stable matching, then at least one agent can profitably misrepresent his/her preferences, assuming the others tell the truth. (This agent can misrepresent in such a way as to be matched to his or her most preferred achievable mate under the true preferences at every stable matching under the false preferences.)

Maybe this speaks negatively about our efforts, but there is also **another aspect**.

The lying has in fact some limits.

Consider a marriage situation with non-empty set of stable matchings $\mathcal{M}$ and **some group of agents lying**. Under the new (false) preferences, the set of stable matchings is $\mathcal{M}'$. The following holds.

Theorem (Demange, Gale, Sotomayor, 1987)

For every μ' in $\mathcal{M}'$, there exists a μ in $\mathcal{M}$ such that at least one liar is not better off under μ' than under μ.

What is good for one... is not necessarily good for the other

We can impose the following relation on stable matchings:

Definition

We say that $\mu \succ_M \mu'$ if all men are at least as satisfied with μ as with μ', with at least one strict preference. (Analogous relation $\succ_W$ can be defined for women.)

This induces a partial ordering $\succ_M$ and $\succ_W$ on stable matchings.

Theorem (Knuth)

When all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings: Let μ and μ' be stable matchings. Then $\mu \succ_M \mu'$ if and only if $\mu' \succ_W \mu$.

In other words: The best outcome for one side of the market is the worst for the other.

Proof of Knuth's lemma

Theorem (restated, Knuth)

When all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings: Let μ and μ' be stable matchings. Then $\mu \succ_M \mu'$ if and only if $\mu' \succ_W \mu$.

Proof.

- Let us prove the "only if" part by means of contradiction. The other part can be argued analogously.
- We have that $\mu \succ_M \mu'$ and $\mu' \not\succ_W \mu$.
- Matchings μ and μ' cannot be the same as we need at least one strict preference.
- Then there is a woman w such that $m = \mu(w) \succ_w \mu'(w)$. Combining with the man m who has clearly a strict preference in μ versus μ', i.e. $w = \mu(m) \succ_m \mu'(m)$, they form a blocking pair in μ', contradicting than μ' is stable.

$\square$

Obviously, there can be a lot of stable matchings. Can we capture the set of stable matching by some structure?

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Yes! They form a **lattice** (a partially ordered set in which every two elements have a unique common supremum and infimum).

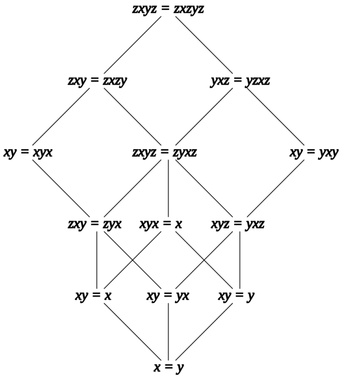

Figure: An example of lattice.

Definition (Meet and join on stable matchings)

The function $\wedge_M$ takes two matchings as their input and for each man and woman, it chooses the better (or the worse, respectively) of the matches for such player. The function $\vee_M$ is defined analogously, but with exchanging men and women.

Theorem (Conway)

Suppose that all preferences are strict. If μ and μ' are both stable matchings, then $\mu \vee_M \mu'$ and $\mu \wedge_M \mu'$ are also stable matchings.

Utilities and connection to cooperative games

The situation and the problem we described involves by its nature **non-transferable utility**. That is because each woman and each man has its own preference relation. Each one of them **values others differently**.

We omit the formal definitions and construction of the game, but matching situations described here can be modelled as special **cooperative games with non-transferable utility** for which the core corresponds exactly to its stable matchings.

One to one/Many to one

So far, the model we have allows us to pair one agent with at most one other agent.

It is useful to consider matching markets in which one (or two) of the sides allows to have more than one connection.

Here we shall outline **many to one** system.

Definition (Workers and firms model)

- Players are **workers**, $W = \{w_i, \ldots, w_n\}$, and **firms**, $F = \{f_1, \ldots, f_p\}$. Each firm f_i has a quota q_i: the number of workers to hire.
- The market is again two-sided.
- Each worker and firm has again a **preference relation** over the respective opposite set.
- Matching is defined analogously, workers are assigned to firms, firm cannot exceed its quota.

The stability can be generalised to this setting with a few technicalities. At first, it seems that not only blocking pair but in general blocking coalitions need to be considered. However, *roughly speaking*, group stability can be proved to be equivalent to pairwise stability. The GS algorithm **can be adapted**.

Two-sided vs. one-sided

Suppose the market is not two-sided, **does a stable matching always exist?**

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Roommate problem in which every agent states preferences on possible roommates and we place them in rooms with either two beds or one bed.

- Agent 1: $2 \succ 3 \succ 1$
- Agent 2: $3 \succ 1 \succ 2$
- Agent 3: $1 \succ 2 \succ 3$

All agents being alone is not a stable matching.

Any matching with two agents in a room is not stable either.

Thank you! Questions?