Complexity in Cooperative Games

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1 Definitions from the first lecture

A *cooperative game* is an ordered pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function of the cooperative game. We further assume that $v(\emptyset) = 0$. The set of all cooperative games with player set N is denoted by G^N.

Subsets of N are called *coalitions* and N itself is called *grand coalition*. We often write v instead of (N, v), because we can easily identify a game with its characteristic function without loss of generality.

To further analyze players' gains, we will need a *payoff vector* which can be interpreted as a proposed distribution of rewards between players. A payoff vector for a cooperative game (N, v) is a vector $x \in \mathbb{R}^N$ with x_i denoting the reward given to the ith player.

An *imputation* of $(N, v) \in G^N$ is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for every $i \in N$. The set of all imputations of a given cooperative game (N, v) is denoted by $I(v)$.

The *core* of $(N, v) \in G^N$ is the set

$$C(v) = \left\{ x \in I(v);\ \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \right\}.$$

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Let $v : 2^N \to \mathbb{R}$ be a game, and let $x \in \mathbb{R}^N$ be a payoff vector. The *excess* of x for a coalition $S \subseteq N$ is the quantity $v(S) - \sum_{i \in S} x_i$; that is, the gain that players in coalition S can obtain if they withdraw from the grand coalition N under payoff x and instead take the payoff $v(S)$.

Now let $\theta(x) \in \mathbb{R}^{2^N}$ be the vector of excesses of x, arranged in non-increasing order. In other words, $\theta_i(x) \geq \theta_j(x), \forall\ i < j$. Notice that x is in the core of v if and only if it is a pre-imputation and $\theta_1(x) \leq 0$. To define the nucleolus, we consider the lexicographic ordering of vectors in $\mathbb{R}^{2^N}$: For two payoff vectors x, y, we say $\theta(x)$ is lexicographically smaller than $\theta(y)$ if for some index k, we have $\theta_i(x) = \theta_i(y), \forall\ i < k$ and $\theta_k(x) < \theta_k(y)$. (The ordering is called lexicographic because it mimics alphabetical ordering used to arrange words in a dictionary.) The *nucleolus* of v is the lexicographically minimal imputation, based on this ordering. This solution concept was first introduced in by Schmeidler (1969).

A *Shapley value* is a function $\phi : G^N \to \mathbb{R}^n$ defined by the following axioms:

- Efficiency: The total gain is distributed: :$\sum_{i \in N} \phi_i(v) = v(N)$
- Symmetry: If i and j are two actors who are equivalent in the sense that :$v(S \cup \{i\}) = v(S \cup \{j\})$ for every subset S of N which contains neither i nor j, then $\phi_i(v) = \phi_j(v)$.
- Linearity: If two coalition games described by gain functions v and w are combined, then the distributed gains should correspond to the gains derived from v and the gains derived from w: :$\phi_i(v+w) = \phi_i(v) + \phi_i(w)$ for every i in N. Also, for any real number a, :$\phi_i(av) = a\phi_i(v)$ for every i in N.
- Zero player (null player): The Shapley value $\phi_i(v)$ of a null player i in a game v is zero. A player i is *null* in v if $v(S \cup \{i\}) = v(S)$ for all coalitions S.

Shapley proved that this function is well-defined, that it is unique and that it has the following form:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\ (n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S)).$$

2 Outline of the second lecture

I will show three complexity results on three different cooperative game models.

- Weighted graph model (wGG) and Shapley value,
- rule-based representation and Shapley value,
- minimum coloring games (MCG) and its core - largeness, exactness.

3 References

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