Homeworks for Discrete and Continuous Optimization

(Milan Hladík, May 22, 2020)

1 Introduction. [deadline 15.4.] **52**

Ex. 1.1 Which cuboid has maximal volume given a fixed surface area S? **6**

Ex. 1.2 The hill has the shape of a cone, where the base is a circle with radius r and the side edge has length s. We are situated at the bottom. Find the shortest roundtrip around the hill such that we terminate at the same position we are situated. (We do not ask to compute its length, but rather a trajectory.) **8**

Ex. 1.3 Solve the optimization problem using the Euclidean norm

$$\max\ c^T x \ \text{ subject to } \ \|x\|_2 \le 1,\ x \in \mathbb{R}^n. \qquad \mathbf{10}$$

(*Hint:* You can guess the optimal solution and then prove its optimality.)

Ex. 1.4 (Generalization of the above exercise.)
Solve the optimization problem with $A \in \mathbb{R}^{n\times n}$ positive definite

$$\max\ c^T x \ \text{ subject to } \ x^T A x \le 1,\ x \in \mathbb{R}^n. \qquad \mathbf{10}$$

Ex. 1.5 Let two sets of points $x_1, \dots, x_p \in \mathbb{R}^n$ and $y_1, \dots, y_q \in \mathbb{R}^n$ be given.

(a) Formulate the problem of finding a (strictly) separating hyperplane as a feasibility problem. **5**

(b) Formulate the problem of finding a widest separating band as an optimization problem. **5**

Ex. 1.6 Let $A = e$ and let $b \in \mathbb{R}^m$ be arbitrary (here, $e = (1, \dots, 1)^T$ is the vector of ones). Solve the linear regression problem using ℓ_p-norm successively for $p = 1, 2, \infty$. Interpret your results. **8**

2 Unconstrained optimization, convex sets. [deadline 15.4.] **52**

Ex. 2.1 Let $x_1, \dots, x_m \in \mathbb{R}$. Find the optimal solution (not the optimal value) of the optimization problems

(a) $\min_{a\in\mathbb{R}} \sum_{i=1}^m (x_i - a)^2$, **6**

(b) $\min_{a\in\mathbb{R}} \sum_{i=1}^m |x_i - a|$. **6**

Ex. 2.2 Let $C \in \mathbb{R}^{n\times n}$ symmetric and nonsingular. Consider the optimization problem

$$\min_{x\in\mathbb{R}^n}\ x^T C x + d^T x.$$

(a) Analyse when the problem has the unique optimal solution, more optimal solutions, and no optimal solution. In the first case, find the optimal solution. In the second case, describe all optimal solutions. In the third case, find an unbounded direction (i.e., a direction in which the objective function tends to $-\infty$). **10**

(b) Let x^* be an optimal solution. Compute the optimal value in time $\mathcal{O}(n)$. **6**
(*Hint:* Elaborate with the optimality condition.)

Ex. 2.3 Let $A, B \subseteq \mathbb{R}^n$ be convex sets. Is the Minkowski sum $A + B$ also convex? **6**

What about the converse direction? **2**

Ex. 2.4 Let $P \in \mathbb{R}^{n\times n}$ be positive definite and $x^0 \in \mathbb{R}^n$. Prove that the ellipsoid $\{x \in \mathbb{R}^n;\ (x - x^0)^T P(x - x^0) \le 1\}$ centered at x^0 is a convex set. **6**

Ex. 2.5 A point $x \in M$ is an extreme point of set $M \subseteq \mathbb{R}^n$ if it is a convex combination of no $y, z \in M \setminus \{x\}$.

(a) Find all extreme points of a square, a disc, a circular sector and an unbounded icecream cone. **6**

(b) Let M be a convex set. Prove that x is an extreme point if and only if $M \setminus \{x\}$ is convex. **4**

3 Convex functions. [deadline 21.4. at 10:30] 52

Ex. 3.1 Show that a function that is both convex and concave is affine then. **6**

Ex. 3.2 Prove the first order characterization of a strictly convex function. **8**

Ex. 3.3 Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be convex and increasing. What you can say about f^{-1}? **8**

Ex. 3.4 Let $f_1, \dots, f_k$ be convex functions on a convex set $M \subseteq \mathbb{R}^n$. Prove that $\max_{i=1,\dots,k} f_i(x)$ is a convex function on M. **4**

Ex. 3.5 Show convexity of the following functions:

(a) $\dfrac{x_1^2}{x_2}$ on the set $\{x \in \mathbb{R}^2;\ x_2 > 0\}$, **8**

(b) $d_M(x) := \inf_{y\in M} \|x-y\|$, that is, the distance of a point to a convex set $M \subseteq \mathbb{R}^n$, **8**

(c) $\log(\sum_{i=1}^n e^{x_i})$ on the set $x \in \mathbb{R}^n$. **10**

4 Convex optimization. [deadline 28.4. at 10:30] 18

Ex. 4.1 Let $\emptyset \ne M \subseteq \mathbb{R}^n$ be convex and suppose that function $f\colon M \to \mathbb{R}$ is convex and attains its maximum in an interior point of M. Prove that $f(x)$ is constant on M then. **6**

Ex. 4.2 Suppose that the system $Ax = b$ has infinitely many solutions and consider the optimization problem

$$\min\ \|x\|_p \ \text{ subject to } \ Ax = b.$$

Depending on $p \in \{1, 2, \infty\}$ discuss the existence and uniqueness of an optimal solution. In case of multiple optima, discuss what is the shape of the optimal solution set and if it is bounded. (We do not ask to determine the optimal solution.) **6**

Ex. 4.3 Denote by $B(c, r)$ the Euclidean ball with center $c \in \mathbb{R}^n$ and radius $r > 0$. Formulate as an optimization problem the problem of finding the smallest ball covering m given balls $B(c_i, r_i)$, $i = 1, \dots, m$. Is it a convex optimization problem? **6**

5 Quadratic programming. [deadline 5.5. at 10:30] **32**

Ex. 5.1 Consider the problem of computing the (minimum) distance between two polyhedra described by $Ax \le b$ and $Cx \le d$, respectively. Formulate it as an optimization problem and classify it. **6**

Ex. 5.2 Reformulate the convex quadratic programming problem

$$\min\ (c^T x)^2 \ \text{ subject to } \ Ax \le b$$

as a linear program. **6**

Ex. 5.3 For given $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ solve the optimization problem

$$\min\ \textstyle\sum_{i=1}^n x_i^2 \ \text{ subject to } \ a^T x \ge b,\ x \ge 0.$$ **10**

Ex. 5.4 Let $C \in \mathbb{R}^{n\times n}$ be positive definite. Show that the problem

$$\min_{x\in M}\ x^T C x + d^T x$$

is equivalent to

$$\min_{x\in M}\ (x + \tfrac{1}{2}C^{-1}d)^T C (x + \tfrac{1}{2}C^{-1}d).$$ **10**

6 Convex cone programming. [deadline 12.5. at 10:30] **52**

Ex. 6.1 In space $\mathbb{R}^n$, find the dual cones to $\mathbb{R}^n$, $\{0\}$, axis x_1, and the positive half-axis of x_1. **6**

Ex. 6.2 Prove:

(a) $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$, **4**

(b) $\mathcal{L}^* = \mathcal{L}$ (the Lorentz cone). **8**

Ex. 6.3 Decide whether the generalized Lorentz cone $\mathcal{L} = \{x \in \mathbb{R}^n;\ x_n \ge \|(x_1,\dots,x_{n-1})\|\}$ is self-dual for the ℓ_1-norm $\|\cdot\|_1$ and maximum norm $\|\cdot\|_\infty$ defined as

$$\|v\|_1 = \textstyle\sum_i |v_i|, \quad \|v\|_\infty = \max_i |v_i|.$$ **10**

Ex. 6.4 Construct the dual problem to the dual problem and compare to the primal problem. **6**

Ex. 6.5 Express as second order cone constraints:

(a) $x^T Q x \le t^2$, $t \ge 0$, where Q is a positive semidefinite matrix, **6**

(b) $\sqrt{xy} \ge t \ge 0$, $x, y \ge 0$, **6**

(c) $x^4 \le t$. **6**

7 KKT conditions. [deadline 19.5. at 10:30] **28**

Ex. 7.1 Apply KKT conditions to a linear program min $c^T x$ subject to $Ax \le b$, and discuss the assumptions. **6**

Ex. 7.2 Write and discuss KKT conditions for a quadratic program

$$\min\ x^T C x \ \text{ subject to } \ Ax \le b.$$ **6**

Ex. 7.3 Solve Ex. 1.1. by using KKT conditions. **6**

Ex. 7.4 By using KKT conditions solve the following problem with given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$

$$\max\ \sum_{i=1}^{n} \log(x_i) \ \text{ subject to } \ a^T x \le b,\ x \ge 0.$$ **10**

8 Methods. [deadline 26.5. at 10:30] **36**

Ex. 8.1 Approximate function $f(x)$ by a quadratic function $q(x)$ such that they have the same:

(a) function values and derivatives at point x_k and the second derivatives at point x_{k-1}, **6**

(b) function values at point x_k and derivatives at points x_k, x_{k-1}. **6**

Ex. 8.2 Consider the problem

$$\min\ x^2 + xy + y^2 - 2y \ \text{ subject to } \ x + y = 2.$$

(a) Solve the problem directly. **2**

(b) Solve the problem analytically by the penalty method and the square function as the penalty function. What is the central path?
(That is, find the otimal solution depending on the penalty parameter and compute its limit.) **6**

Ex. 8.3 By the penalty method solve the problem

$$\min\ x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1 + 5x_2 - 6x_3 \ \text{ subject to } \ x_1^2 + x_2^2 + x_3^2 = 100.$$

Take the initial point $x = (0, 0, 0)^T$ and the square function as the penalty function. Draw the central path approximation in the subspace of coordinates (x_1, x_2). For solving the auxiliary (unconstrained) sub-problems, you can use a suitable software (Matlab, Octave, Maple, ...) **8**

Ex. 8.4 By the barrier method solve the problem

$$\min\ x_1^2 + 2x_2^2 + 3x_3^2 - 14x_1 + 15x_2 - 16x_3 \ \text{ subject to } \ x_1^2 + x_2^2 + x_3^2 \le 100.$$

Take the initial point $x = (0, 0, 0)^T$ and the logarithmic barrier function. Draw the central path approximation in the subspace of coordinates (x_1, x_2). For solving the auxiliary (unconstrained) sub-problems, you can use a suitable software (Matlab, Octave, Maple, ...) **8**