Exercise 11: LP & CO

1. Give cutting plane proofs for the following.

   a. Let $G = (V, E)$ be a graph. Prove that the inequalities $\sum_{e \in E[U]} x_e \leqslant \frac{|U| - 1}{2}$ for $U \subseteq V$ a set of odd cardinality, are Chvátal-Gomory cutting planes with respect to the system:

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &\leqslant 1, && \forall v \in V \\ x_e &\geqslant 0, && \forall e \in E \\ x_e &\in \mathbb{Z}, && \forall e \in E \end{aligned}$$

   b. Let $G = (V, E)$ be a graph. Prove that the inequalities $\sum_{e \in \delta(U)} x_e \geqslant 1$ for $U \subseteq V$ a set of odd cardinality, are Chvátal-Gomory cutting planes with respect to the system:

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &= 1, && \forall v \in V \\ x_e &\geqslant 0, && \forall e \in E \\ x_e &\in \mathbb{Z}, && \forall e \in E \end{aligned}$$

   c. Let $G = (V, E)$ be a graph. Prove that the inequalities $\sum_{e \in \delta(S)} x_e \geqslant 2$ are Chvátal-Gomory cutting planes with respect to the system:

$$\begin{aligned} \sum_{e \in \delta(U)} x_e &\geqslant 1, && \forall \emptyset \neq U \subsetneq V \\ 0 \leqslant x_e &\leqslant 1, && \forall e \in E \\ x_e &\in \mathbb{Z}, && \forall e \in E \end{aligned}$$

   d. Let $G = (V, E)$ be a graph and let STAB(G) be the convex hull of all independent sets in G. Let C be any odd cycle in G. Prove that the inequality $\sum_{v \in V(C)} x_v \leqslant \frac{|C| - 1}{2}$ are Chvátal-Gomory cutting planes with respect to the system:

$$\begin{aligned} x_u + x_v &\leqslant 1, && \forall \{u, v\} \in E \\ x_v &\geqslant 0, && \forall v \in V \\ x_v &\in \mathbb{Z}, && \forall v \in V \end{aligned}$$

   f. Let $G = (V, E)$ be a graph and let STAB(G) be the convex hull of all independent sets in G. Let K be any clique in G. Prove that the inequality $\sum_{v \in V(K)} x_v \leqslant 1$ are Chvátal-Gomory cutting planes with respect to the system:

$$\begin{aligned} x_u + x_v &\leqslant 1, && \forall \{u, v\} \in E \\ x_v &\geqslant 0, && \forall v \in V \\ x_v &\in \mathbb{Z}, && \forall v \in V \end{aligned}$$