# Exercise 10: LP & CO

1. Let $G = (V, E)$ be a graph and let $\mathcal{I} = \{F \subseteq E \mid F \text{ is a matching in } G\}$. Give an example where $(E, \mathcal{I})$ is not a matroid.

2. Let $M = (E, \mathcal{I})$ be a matroid, and let $E^{'} \subseteq E$. Define $\mathcal{I}^{'} = \{I \cap E^{'} \mid I \in \mathcal{I}\}$. Prove that $(E^{'}, \mathcal{I}^{'})$ is a matroid.

3. Let $M = (E, \mathcal{I})$ be a matroid, and $X \in \mathcal{I}$. Let $X' \supseteq X$ be an inclusion-wise maximal independent set. Prove that $|X'| = r_M(E)$, where $r_M(\cdot)$ is the rank function of $M$.

4. Using the description of matroid polytopes, prove that the spanning tree polytope of a graph has the following description

$$\left\{ x \;\middle|\; x \geqslant 0, \sum_{e \in S} x_e \leqslant |S| - 1 \;\; \forall S \subseteq V \right\}.$$

5. Prove that the following two definitions of a Matroid are equivalent.

   **D1.** A matroid M is a pair $(E, \mathcal{I})$ consisting of a finite set $E$ and $\mathcal{I} \subseteq 2^E$ satisfying:
   (a) $\emptyset \in \mathcal{I}$.
   (b) Let $X \in \mathcal{I}$. Then, for all $Y \subset X, Y \in \mathcal{I}$.
   (c) Let $X, Y \in \mathcal{I}$ and $|Y| > |X|$. Then, there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$.

   **D2.** A matroid M is a pair $(E, \mathcal{I})$ consisting of a finite set $E$ and $\mathcal{I} \subseteq 2^E$ satisfying:
   (a) $\emptyset \in \mathcal{I}$.
   (b) Let $X \in \mathcal{I}$. Then, for all $Y \subset X, Y \in \mathcal{I}$.
   (c) Let $\mathcal{B}$ be the set of inclusion-wise maximal elements of $\mathcal{I}$. Let $X, Y \in \mathcal{B}$ be distinct and let $x \in X$. Then,there exists $y \in Y$ such that $X \setminus \{x\} \cup \{y\} \in \mathcal{B}$.