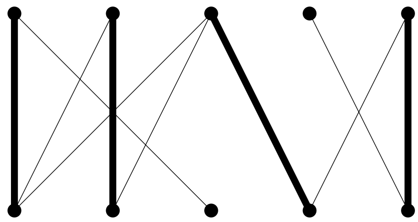
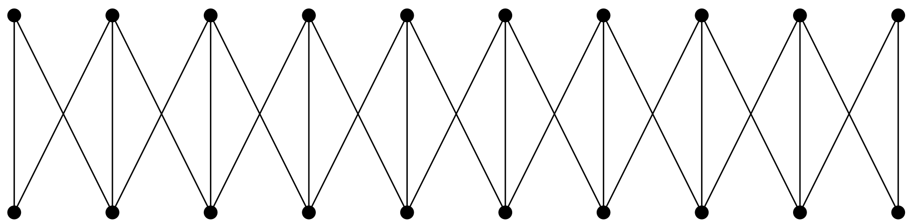
Exercise 8: LP & CO

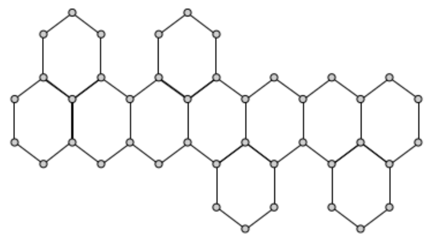
1. Your “friend” claims that she has found the largest matching for the graph below (her matching is in bold). She explains that no other edge can be added, because all the edges not used in her partial matching are connected to matched vertices. Is she correct?



2. Count the number of perfect matchings in the following graph.



3. Prove that the following graph does not contain a perfect matching.



4. Consider the game *Slither* by S. Anderson. Given an undirected, simple, connected graph $G = (V, E)$, two players take turns to choose an edge e with the following rule. The edge e is not currently chosen, and the set of edges chosen so far forms a simple path. The player unable to choose an edge loses.

   Prove that the first player has a winning strategy if G has a perfect matching.

5. Let $G = (V, E)$ be a graph with two special nodes $s, t \in V$. A cut in G is defined by a set of vertices $S \subseteq V$ and is the set of edges with exactly one endpoint in S. A cut is called an $s{-}t$ cut if exactly one of s or t is in the set defining the cut. For non-negative edge-weights one can compute a minimimum weight $s{-}t$ cut in polynomial time.

   a. Prove that for non-negative edge weights the minimum cut in any graph can be computed using polynomially many minimum $s{-}t$ cut computations.

   b. What is the smallest polynomial you can obtain in the previous question?

6. Let $G = (V, E)$ be a graph and let $\mathrm{TSP}(G)$ be the convex hull of all hamiltonian cycles in G. Let $\emptyset \neq S \neq V$. Prove that the inequalities $\sum_{e \in \delta(S)} x_e \geqslant 2$ is valid for $\mathrm{TSP}(G)$.

7. Let $G = (V, E)$ be a graph. Consider the following IP formulation for the Traveling Salesman Problem (TSP):

$$\begin{aligned} \sum_{e \ni v} x_e &= 2 & \forall v \in V \\ \sum_{e \in \delta(S)} x_e &\geqslant 2 & \forall \emptyset \neq S \neq V \\ x_e &\in \{0, 1\} & \forall e \in E \end{aligned}$$

where $\delta(S)$ denotes the set of edges with exactly one endpoint in S.

   a. Prove that the above formulation is correct. That is, prove that $\mathbf{x}$ is feasible for the above IP if and only if it is the characteristic vector of a Hamiltonian cycle in G.

   b. Give a polynomial time algorithm that on input $\mathbf{x} \in \mathbb{R}^{|E|}$ accepts $\mathbf{x}$ is it satisfies the LP relaxation of the above IP and outputs a violated constraint otherwise.