Exercise 7: LP & CO

1. Let $G = (V, E)$ be a graph and let M and M' be two matchings in G. Prove that $(V, M \triangle M')$ is a disjoint union of paths and cycles, where $A \triangle B$ is the symmetric difference of the two sets A and B.

2. Let $G = (V, E)$ be a graph and let $U \subset V$ be a set of odd cardinality and let $E[U] = \{\{u, v\} \in E | u, v \in U\}$. Prove that $\sum_{e \in E[U]} x_e \leqslant \frac{|U|-1}{2}$ is a valid inequality for the matching polytope of G.

3. Let $G = (V, E)$ be a graph and let $M, M^{'}$ be two distinct matchings in G.

   a. Give a vector $\mathbf{c} \in \mathbb{R}^{|E|}$ such that M is the unique optimum of the LP: $\max \mathbf{c}^\intercal \mathbf{x}$ s.t. $\mathbf{x} \in \mathrm{PM}(G)$.

   b. Prove that if $M \triangle M^{'}$ contains at least two components (disregarding isolated vertices) then there exist two matchings $M^{''}$ and $M^{'''}$ different from M and $M^{'}$ such that $\chi^{(M)} + \chi^{(M^{'})} = \chi^{(M^{''})} + \chi^{(M^{'''})}$.

   c. Prove that if $M \triangle M^{'}$ is a single component (disregarding isolated vertices) then there exists a vector $\mathbf{c} \in \mathbb{R}^{|E|}$ such that $\mathrm{conv}(\{M, M^{'}\})$ is the set of optimal solutions of the LP: $\max \mathbf{c}^\intercal \mathbf{x}$ s.t. $\mathbf{x} \in \mathrm{PM}(G)$.

4. Prove that a tree can have at most one perfect matching.

5. Let G be a graph and M_1, M_2 two maximal matchings in G. Prove that $|M_1| \leqslant 2 \cdot |M_2|$.

*6 Count the number of perfect matchings in the following graph.