Exercise 6: LP & CO

1. Write the dual for following LPs:

   (a)

$$
\begin{array}{lrrrrcr}
\min & 3x_1 & +2x_2 & -3x_3 & +4x_4 & & \\
\\
\text{s.t.} & x_1 & -2x_2 & +3x_3 & +4x_4 & \leqslant & 3 \\
 & & x_2 & +3x_3 & +4x_4 & \geqslant & -5 \\
 & 2x_1 & -3x_2 & -7x_3 & -4x_4 & = & 2 \\
\\
 & x_1, & x_2, & x_3, & x_4, & \geqslant & 0
\end{array}
$$

   (b)

$$
\begin{array}{lrrcr}
\max & 3x_1 & +2x_2 & & \\
\\
\text{s.t.} & x_1 & +3x_2 & \leqslant & 3 \\
 & 6x_1 & -x_2 & = & 4 \\
 & x_1 & +2x_2 & \leqslant & 2 \\
\\
 & x_1, & x_2 & \geqslant & 0
\end{array}
$$

2. Consider the following LP:

$$
\begin{array}{lrrrrcr}
\max & 2x_1 & +x_2 & +3x_3 & +x_4 & & \\
\\
\text{s.t.} & x_1 & +x_2 & +x_3 & +x_4 & \leqslant & 5 \\
 & 2x_1 & -x_2 & +3x_3 & & = & -4 \\
 & x_1 & & -x_3 & +x_4 & \geqslant & 1 \\
\\
 & x_1 & & & & \geqslant & 0 \\
 & & x_2 & & & & \text{free} \\
 & & & x_3 & & \geqslant & 0 \\
 & & & & x_4 & & \text{free}
\end{array}
$$

   (a) State the problem in equational form.
   (b) Write the dual of the given problem and the transformed problem.
   (c) Show that these duals are equivalent.

3. Consider the following linear program:

$$
\begin{array}{lrrrrrcr}
\max & & -4x_2 & +3x_3 & +2x_4 & -8x_5 & & \\
\\
\text{s.t.} & 3x_1 & +x_2 & +2x_3 & +x_4 & & = & 3 \\
 & x_1 & -x_2 & & +x_4 & -x_5 & \geqslant & 2 \\
\\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geqslant & 0
\end{array}
$$

   (a) Write the dual LP.
   (b) Solve the dual LP graphically.

(c) Use complimentary slackness to determine which primal variables are zero in the optimum.

(d) Find the optimum primal solution.

4. Show that a matrix A is totally unimodular if and only if the matrix $[A \mid -A \mid I \mid -I]$ is totally unimodular, where $[A \mid B]$ is the matrix obtained by concatinating matrix B to A along columns.

5. An interval matrix is a matrix $M \in \{0,1\}^{m \times n}$ such that in each row the ones appear as a consecutive block. Prove that interval matrices are totally unimodular. (Hint: you may find elementary column operations useful.)