Exercise 5: LP & CO

Problems marked with * are homeworks. Solutions must be submitted by 11.04.2024 15:39 pm using postal owl.

1. Let P be the (primal) LP: $\max c^\intercal x$ s.t. $Ax \leqslant b, x \geqslant 0$, and let D be its dual LP. Prove that the dual of D is P.

2. Write the dual for each of the following forms of LP (For P_2, P_3 derive them by first converting them to the standard form P_1):

$\mathbf{P}_1$: $\max\ \boldsymbol{c}^\intercal \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leqslant b$, $\boldsymbol{x} \geqslant 0$	$\mathbf{P}_2$: $\max\ \boldsymbol{c}^\intercal \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leqslant b$	$\mathbf{P}_3$: $\max\ \boldsymbol{c}^\intercal \boldsymbol{x}$ s.t. $A\boldsymbol{x} = b$, $\boldsymbol{x} \geqslant 0$

3. Consider the following LP:

$$
\begin{array}{llll}
\text{P:} & \max & 2x_1 + 3x_2 & \\
 & \text{s.t.} & 4x_1 + 8x_2 & \leqslant 12 \\
 & & 2x_1 + x_2 & \leqslant 3 \\
 & & 3x_1 + 2x_2 & \leqslant 4 \\
 & & x_1, x_2 & \geqslant 0
\end{array}
$$

   a. Write the dual LP.

   b. Solve P and the dual by the Simplex method.

4. Consider the following LP:

$$
\begin{array}{llllll}
\max & \boldsymbol{c}_1^\intercal \boldsymbol{x} & \boldsymbol{c}_2^\intercal \boldsymbol{y} & \boldsymbol{c}_3^\intercal \boldsymbol{z} & \\
\text{s.t.} & \boldsymbol{A}_{11}\boldsymbol{x} & +\boldsymbol{A}_{12}\boldsymbol{y} & +\boldsymbol{A}_{13}\boldsymbol{z} & = \boldsymbol{b}_1 \\
 & \boldsymbol{A}_{21}\boldsymbol{x} & +\boldsymbol{A}_{22}\boldsymbol{y} & +\boldsymbol{A}_{23}\boldsymbol{z} & \geqslant \boldsymbol{b}_2 \\
 & \boldsymbol{A}_{31}\boldsymbol{x} & +\boldsymbol{A}_{32}\boldsymbol{y} & +\boldsymbol{A}_{33}\boldsymbol{z} & \leqslant \boldsymbol{b}_3 \\
 & \boldsymbol{x} & & & \geqslant \boldsymbol{0} \\
 & & \boldsymbol{y} & & \leqslant \boldsymbol{0} \\
 & & & \boldsymbol{z} & \text{free}
\end{array}
$$

   *(a) Write the dual LP.

   (b) Let $(\boldsymbol{x}^\intercal, \boldsymbol{y}^\intercal, \boldsymbol{z}^\intercal)^\intercal$ and $(\boldsymbol{u}^\intercal, \boldsymbol{v}^\intercal, \boldsymbol{w}^\intercal)^\intercal$ be arbitrary primal and dual feasible solutions. Prove that

$$\boldsymbol{b}_1^\intercal \boldsymbol{u} + \boldsymbol{b}_2^\intercal \boldsymbol{v} + \boldsymbol{b}_3^\intercal \boldsymbol{w} \leqslant \boldsymbol{c}_1^\intercal \boldsymbol{x} + \boldsymbol{c}_2^\intercal \boldsymbol{y} + \boldsymbol{c}_3^\intercal \boldsymbol{z}.$$

5. (*Complimentary slackness*) Let P be the (primal) LP: $\max c^\intercal x \text{s.t.} Ax \leqslant b, x \geqslant 0$, and let D be its dual LP. Let x, y be their respective optimal solution. Then prove that:

   a. $x_i > 0 \implies$ the i-th constraint in D in satisfied with equality.

   b. $y_j > 0 \implies$ the j-th constraint in P is satisfied with equality.

6. Let $P = \{x | A\boldsymbol{x} \leqslant b, \boldsymbol{x} \geqslant 0\}$ be a polyhedron and let $\alpha^\intercal \boldsymbol{x} \leqslant \beta$ be a valid inequality for P. Consider the following two LPs:

$\mathbf{P}_1$: $\max\ \boldsymbol{c}^\intercal \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leqslant b$, $\boldsymbol{x} \geqslant 0$	$\mathbf{P}_2$: $\max\ \boldsymbol{c}^\intercal \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leqslant b$, $\alpha^\intercal \boldsymbol{x} \leqslant \beta$, $\boldsymbol{x} \geqslant 0$

a. Prove that P_1 is equivalent to P_2. That is, $\boldsymbol{x}$ is an optimal solution of P_1 iff it is an optimal solution of P_2.

b. Write the dual LP for each of the two programs above.

c. Are the two dual LPs equivalent as well?