Exercise 3: LP & CO

1. A polyhedron is defined to be a set obtained by intersecting finitely many halfspaces. An inequality $a^\intercal x \leqslant b$ is said to be valid for a polyhedron P if for all $y \in P$ we have that $a^\intercal y \leqslant b$. A face of a polyhedron P is defined to be a set obtained by intersecting P with $\{x \mid a^\intercal x = b\}$ for some inequality $a^\intercal x \leqslant b$ that is valid for P. A bounded polyhedron is called a polytope. A point $v \in P$ called an extreme point of the polyhedron $P \subseteq \mathbb{R}^n$ if there exists a nonzero vector $a \in \mathbb{R}^n$ such that $\forall x \in P : x \neq v \implies a^\intercal x < a^\intercal v$.

Let P be a polytope.

a. Prove that the empty set $\emptyset$ and P are both faces of P.

b. Prove that any face of a polytope is also a polytope.

c. Prove that every polytope $P \neq \emptyset$ has at least one extreme point.

d. Let F be a face of P and let F' be a face of F. Prove that F' is a face of P as well.
To this end, Let $F = \{x \mid a_1^\intercal x = b_1\} \cap P$ and $F' = \{x \mid a_2^\intercal x = b_2\} \cap F$ where $a_1^\intercal x \leqslant b_1$ is valid for P and $a_2^\intercal x \leqslant b_2$ is valid for F. Define $G = \{x \mid N a_1^\intercal x + a_2^\intercal x = N b_1 + b_2\} \cap P$ where $N \in \mathbb{R}_+$.

i. Prove that $F' \subseteq G$.
ii. Prove that $G \subseteq F'$ for sufficiently large N.
iii. Prove that $N a_1^\intercal x + a_2^\intercal x \leqslant N b_1 + b_2$ is valid for P for sufficiently large N.
iv. Using the above results, prove that F' is a face of P.

2. Let P be a polytope and let $\mathcal{F}(P)$ be the set of its faces.

a. Prove that $(\mathcal{F}(P), \subseteq)$ is a poset.

b. Draw the Hasse Diagram for $(\mathcal{F}(C_n), \subseteq)$ where C_n is a polygon with n vertices.

3. A polyhedron is called pointed if it has at least one extreme point.

a. Prove that a polyhedron is pointed if and only if it does not contain a line.

b. Prove that any polyhedron P defined as $Ax = b, x \geqslant 0$ is a pointed polyhedron.

c. Prove that if the value of $c^\intercal x$ is bounded above for feasible points of $P \neq \emptyset$ described in the form above, then there exists an extreme point of P that attains this maximum value.

4. Let $S \neq \emptyset$ be a closed convex subset of the Euclidean space $\mathbb{R}^n$ and let $x \notin S$. Prove that there exists nonzero vector $a \in \mathbb{R}^n$ and a nonzero real number c such that $a^\intercal x < c$ and for all $y \in S$, $a^\intercal y > c$. (If you do not see an idea of how to prove such a thing after 5 minutes of thought, you can read the Wikipedia article on "Hyperplane separation theorem").

5. Consider three classes of problems $\mathrm{LP}_1, \mathrm{LP}_2$, and LP_3 defined by (general) instances as below:

$\mathbf{LP}_1$:	max	$\mathbf{c}^\intercal \mathbf{x}$	$\mathbf{LP}_2$:	max	$\mathbf{c}^\intercal \mathbf{x}$	$\mathbf{LP}_3$:	max	$\mathbf{c}^\intercal \mathbf{x}$
	s.t.	$A_1 \mathbf{x} \leqslant b_1$		s.t.	$A_2 \mathbf{x} \leqslant b_2$		s.t.	$A_3 \mathbf{x} = b_3$
					$\mathbf{x} \geqslant 0$			$\mathbf{x} \geqslant 0$

a. Prove that if there exists an algorithm to solve an arbitrary instance of LP_3 in time polynomial in the size of the input matrix A_3, then there exists an algorithm to solve an abtitrary instance of LP_1 in time polynomial in the size of the input matrix A_1 as well.

b. Prove that if there exists an algorithm to solve an arbitrary instance of LP_3 is time polynomial in the size of the input matrix A_3, then there exists an algorithm to solve an abtitrary instance of LP_2 in time polynomial in the size of the input matrix A_2 as well.

c. Is it okay to ignore the length of the vectors b_i and the number of variables in the above questions?