Exercise 2: LP & CO

1. Let $x_1, x_2, x_3 \in \{0, 1\}$. Write a linear constraint enforcing the condition that not all of them are zero.

2. Let Φ be a 3-CNF formula with n variables and m clauses.

- **a.** Write an Integer Program for deciding whether or not Φ is satisfiable.
- **b.** What is the size of your IP? Try to make it as small as you can.
- **c.** Can you make a plausibility argument as to why you can't make your IP using $o(n/\log n)$ variables? (Hint: Look up Exponential Time Hypothesis (ETH) and LLL algorithm for IP)

3. Prove that the following integer programs can be equivalently reformulated as 0/1 integer programs. Try to use as few extra variables as you can.

- **a.**

$$\begin{aligned} \max\ & c^\intercal x \\ \text{s.t. } & Ax \leqslant b \\ & x_i \in S \end{aligned}$$

  Where S is an arbitrary set of real numbers and $|S| = k$.

- **b.**

$$\begin{aligned} \max\ & c^\intercal x \\ \text{s.t. } & Ax \leqslant b \\ & 0 \leqslant x_i \leqslant M \\ & x_i \in \mathbb{Z} \end{aligned}$$

4. Prove that if x^* is an optimal solution of the LP relaxation of an IP then, if x^* is integral, it is also an optimal soltution of the IP.

5. A matching M in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that for each vertex in V at most one incident edge is selected in M (Note the slight difference from perfect matching). Given $G = (V, E)$ and a cost on edged $c : E \to \mathbb{R}$ we would like to compute a matching M of maximum weight.

- **a.** Prove that we can assume without loss of generality that $c(e) > 0$ for all $e \in E$.
- **b.** Write an IP formulation for the problem. That is, write an IP whose optimal solution gives the desired answer and the feasible region is exactly the set of all matchings in G.
- **c.** Prove that your formulation is correct.
- **d.** Prove that if G is bipartite then the LP relaxation of your IP always has an integral optimum.

6. In the lecture we saw an IP formulation for the unweighted version of the minimum vertex cover problem. We want to reprove those results for the weighted case. To this end, let $G = (V, E)$ be a graph and let $c : V \to \mathbb{R}$ be function.

- **a.** Argue that, without loss of generality, we may assume that $c(v) > 0$ for all $v \in V$.
- **b.** Modify the IP discussed in the lecture to find a vertex cover of G that has minimum cost, where the cost of a subset $S \subseteq V$ is given by $c(S) = \sum_{v \in S} c(v)$.

c. Let x^* be an optimum solution of the LP relaxation of the above IP. Define $S_{\text{LP}} = \{v \mid x_v^* \geqslant 1/2\}$.

i. Prove that S_{LP} is a vertex cover of G.

ii. Prove that $c(S_{\text{LP}}) \leqslant 2\text{OPT}$ where OPT denotes the cost of a minimum vertex cover in G.

7. What goes wrong if we use the idea for approximating Vertex Cover to the perfect matching problem? To this end, answer the following:

a. Write an IP formulation for computing the maximum weight perfect matching in a graph G. That is, write an IP whose optimal solution gives the desired answer and the feasible region is exactly the set of all perfect matchings in G.

b. Prove that your formulation is correct.

c. Suppose we instead solve the LP relaxation of the above IP and let x^* be an optimal solution of the relaxation. Consider the integral (0/1) vector $y \in \{0,1\}^{|E|}$ obtained by setting $y_i = 1$ iff $x_i^* \geqslant 1/2$. How bad an answer could y be for the original problem?

8. Practice problems for writing IPs, including solutions and perhaps other goodies for the interested. (`http://www.math.clemson.edu/~mjs/courses/mthsc.440/integer.pdf`)

9. Prove that a graph is bipartitie if and only if it does not contain an odd cycle as a subgraph.