- **1.** A set S is called convex if for every  $x, y \in S$  it holds that  $t \in [0, 1] \implies tx + (1 t)y \in S$ .
  - (a) Prove that the set of feasible solutions of any Linear Program is convex.
  - (b) Prove that if an LP has two distinct optimal solutions then it has infinitely many optimal solutions.
- **2.** Let  $x_1, x_2, x_3 \in \{0, 1\}$ . Write a linear constraint enforcing the condition that not all of them are zero.
- **3.** Let G = (V, E) be a graph. Write an Integer Linear Program that computes the size of the largest independent set of G.
- **4.** Let  $\Phi$  be a 3-CNF formula with *n* variables and *m* clauses.
  - **a.** Write an Integer Program for deciding whether or not  $\Phi$  is satisfiable.
  - **b.** What is the size of your IP? Try to make it as small as you can.
  - c. Can you make a plausibility argument as to why you can't make your IP using  $o(n/\log n)$  variables? (Hint: Look up Exponential Time Hypothesis (ETH) and LLL algorithm for IP)
- (HW) 5. Let  $x^*$  be an optimal solution of the LP relaxation of an IP. Prove that if  $x^*$  is integral then it is also an optimal solution of the IP.
  - 6. Consider three classes of problems  $LP_1, LP_2$ , and  $LP_3$  defined by (general) instances as below:  $LP_1$ : max  $\mathbf{c}^{\mathsf{T}}\mathbf{x}$  $LP_2$ :  $\max \mathbf{c}^{\mathsf{T}}\mathbf{x}$  $LP_3$ : max  $\mathbf{c}^{\mathsf{T}}\mathbf{x}$  $A_1 \mathbf{x} \leq b_1$  $A_2 \mathbf{x} \leqslant b_2$ s.t. s.t. s.t.  $A_3\mathbf{x} = b_3$  $\mathbf{x} \ge 0$  $\mathbf{x} \geqslant 0$ 
    - **a.** Prove that if there exists an algorithm to solve an arbitrary instance of LP<sub>3</sub> in time polynomial in the size of the input matrix  $A_3$ , then there exists an algorithm to solve an abitrary instance of LP<sub>1</sub> in time polynomial in the size of the input matrix  $A_1$  as well.
    - **b.** Prove that if there exists an algorithm to solve an arbitrary instance of  $LP_3$  is time polynomial in the size of the input matrix  $A_3$ , then there exists an algorithm to solve an abitrary instance of  $LP_2$  in time polynomial in the size of the input matrix  $A_2$  as well.
    - **c.** Is it okay to ignore the length of the vectors  $b_i$  and the number of variables in the above questions?
  - 7. Prove that the following integer programs can be equivalently reformulated as 0/1 integer programs. Try to use as few extra variables as you can.
    - a.

$$\max c^{\mathsf{T}} x$$
  
s.t.  $Ax \leq b$   
 $x_i \in S$ 

Where S is an arbitrary set of real numbers and |S| = k.

b.

$$\max c^{\mathsf{T}} x$$
  
s.t.  $Ax \leq b$   
 $0 \leq x_i \leq M$   
 $x_i \in \mathbb{Z}$ 

8. Let  $LP(c_1, c_2)$  denote the following LP:

$$\max c_1 x_1 + c_2 x_2$$

s.t. 
$$\begin{pmatrix} -1 & 0\\ 0 & -1\\ -1 & 1\\ 1 & 6\\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1\\ x_2 \end{pmatrix} \leqslant \begin{pmatrix} 0\\ 0\\ 1\\ 15\\ 10 \end{pmatrix}$$

- (a) Prove that if  $(x_1, x_2)$  is a feasible solution for LP(1, 1) then  $x_2 \leq \frac{16}{7}$  and  $x_1 \leq \frac{11}{3}$ . (Hint: What can we deduce about a feasible solution using the third and fourth constraints?)
- (b) Prove that any optimal solution for LP(1,1) has an objective value of at most 5.
- (c) Prove that any optimal solution for  $LP(\frac{1}{6}, 1)$  has an objective value of at most  $\frac{5}{2}$ .
- (d) Describe the set of all optimal solutions of  $LP(\frac{1}{6}, 1)$ . Derive your result only using algebra and logic (i.e. avoid drawing pictures). (Hint: Consider the set of all feasible points that satisfy the fourth constraint with equality. If this set is non-empty then the set of optimal solutions is exactly this set. Why?)
- **9.** Let LP(c, A, b) be the LP:

$$\max c^{\mathsf{T}}x \text{ s.t. } Ax \leq b,$$

where A is an  $m \times n$  matrix and the other vectors have appropriate sizes. Suppose  $\lambda \in \mathbb{R}^m$  be any vector such that  $A^{\mathsf{T}}\lambda = c; \lambda \ge 0$ . Prove that any feasible solution of LP(c, A, b) (and hence any optimal solution) has the objective value at most  $b^{\mathsf{T}}\lambda$ . (Hint: Use your reasoning from 8. (a)-(c).)