Exercise 1: LP & CO

1. Write the following LP in the standard form:

$$\begin{array}{lrcl} \min & x_1 - x_2 + 3x_3 & & \\ \text{s.t.} & x_1 + x2 + x_3 & = & 5 \\ & x_1 + 4x_2 & \geqslant & 4 \\ & 3x_1 - x_3 & \leqslant & 2 \end{array}$$

2. A set S is called convex if for every $x, y \in S$ it holds that $t \in [0,1] \implies tx + (1-t)y \in S$.

a Prove that the set of feasible solutions of any Linear Program is convex.

b Prove that if an LP has two distinct optimal solutions then it has infinitely many optimal solutions.

3. Let LP(c_1, c_2) denote the following LP:

$$\begin{array}{ll} \max & c_1x_1 + c_2x_2 \\ \text{s.t.} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 1 \\ 1 & 6 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leqslant \begin{pmatrix} 0 \\ 0 \\ 1 \\ 15 \\ 10 \end{pmatrix} \end{array}$$

(a) Prove that if (x_1, x_2) is a feasible solution for LP$(1,1)$ then $x_2 \leqslant \frac{16}{7}$ and $x_1 \leqslant \frac{11}{3}$. (Hint: What can we deduce about a feasible solution using the third and fourth constraints?)

(b) Prove that any optimal solution for LP$(1,1)$ has an objective value of at most 5.

(c) Prove that any optimal solution for LP$(\frac{1}{6}, 1)$ has an objective value of at most $\frac{5}{2}$.

(d) Describe the set of all optimal solutions of LP$(\frac{1}{6}, 1)$. Derive your result only using algebra and logic (i.e. avoid drawing pictures). (Hint: Consider the set of all feasible points that satisfy the fourth constraint with equality. If this set is non-empty then the set of optimal solutions is exactly this set. Why?)

4. Let LP(c, A, b) be the LP:

$$\max\ c^\intercal x \text{ s.t. } Ax \leqslant b,$$

where A is an $m \times n$ matrix and the other vectors have appropriate sizes. Suppose $\lambda \in \mathbb{R}^m$ be any vector such that $A^\intercal \lambda = c; \lambda \geqslant 0$. Prove that any feasible solulion of LP(c, A, b) (and hence any optimal solution) has the objective value at most $b^\intercal \lambda$. (Hint: Use your reasoning from **3.** (a)-(c).)

5. Let $G = (V, E)$ be a graph. Write an Integer Linear Program that computes the size of the largest independet set of G.

6. When does one halfspace contain another? That is, give conditions under which $\{x | a_1^\intercal x \leqslant b_1\} \subseteq \{x | a_2^\intercal x \leqslant b_2\}$.

7. What is the distance between two parallel hyperplanes $\{x | a^\intercal x = b_1\}$ and $\{x | a^\intercal x = b_2\}$?