
Problem 1. *Use induction to prove the following statements:*

1. *(HW)* $\forall n \in \mathbb{N}, 5^n - 1$ *is divisible by* 4.
2. $\forall n \in \mathbb{N}, 2^n \leq (n+1)!$.

Solution.

1. The statement is clearly true for $n = 1$, so we can assume that for n, $5^n = 4m + 1$ for some value of m. Then $5^{n+1} - 1 = 5(4m+1) - 1 = 20m + 5 - 1 = 4(5m+1)$. Finishing the proof.
2. The statment is clearly true for $n = 1$, so we can assume that for n, $2^n \leq (n+1)!$. We want to show that $2^{n+1} \leq (n+2)!$. But $2^{n+1} = 2 \cdot 2^n$ and $(n+2)! = (n+2) \cdot (n+1)!$. Clearly $2 < n+2$ and by inductive hypothesis $2^n \leq (n+1)!$, so the result follows.

□

Problem 2. *Let a relation* $\sqsubseteq$ *on* $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ *be defined as* $(a, b) \sqsubseteq (c, d)$ *if* $ad \leq cb$. *Is* $\sqsubseteq$ *a partial order?*

Solution. It is not a partial order, antysimmetry fails for $(4, 2)$ and $(6, 3)$. □

Problem 3. *Let* $a_1, ..., a_n$ *be* n *integers which are not necessarily distinct. Prove that there is always a set of consecutive numbers* $a_k, a_{k+1}, ..., a_l$ *whose sum is a multiple of* n. *Hint: Define a function sending* $m \in \{1, ..., n\}$ *to the remainder of* $\sum_{i=1}^{m} a_i$ *when divided by* n. *Then use pigeonhole principle.*

Solution. Define a function f from $\{1, ..., n\}$ to $\{0, 1, ..., n-1\}$ by sending m to the remainder of $\sum_{i=1}^{m} a_i$ when divided by n. Then it's clear that there are k, l such that $f(k) = f(l)$. Then $\sum_{i=k+1}^{l} a_i$ has remainder 0 when divided by n. □

Problem 4. *Prove that* $E[X^2] \geq (E[X])^2$ *holds for any random variable* X.

Solution. Calculate the expected value $E((X - E(X))^2) = E(X^2 - 2XE(X) + E(X)^2) = E(X^2) - E(X)^2$. Since $(X - E(X))^2$ is a positive random variable, it's expected value is positive. The result follows. □

Problem 5. *Euler's formula* ($v - e + f = 2$) *holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of* $v - e + f$ *now? What if it has* k *components?*

Solution. The formula is $v - e + f = k + 1$. For each component we have $v_i - e_i + f_i = 2 \implies v - e + f = 2k$. But, the external face has been counted k times, once for each component so we get the actual solution by substracting the $k - 1$ additional counts. □

Problem 6. *Let R be a relation over a set X. The symmetric closure of R is the relation $R \cup R^{-1}$. The transitive closure of R is the smallest superset of R that is transitive.*

1. *Prove that the symmetric closure of R is the smallest superset of R that is symmetric.*
2. *Prove that the transitive closure of R is $\bigcup_{i=1}^{\infty} R^i$, where $R^1 = R$ and $R^{i+1} = R \circ R^i$.*
3. *Prove that the transitive closure of a symmetric relation is symmetric.*
4. *Prove that the symmetric closure of a transitive relation need not be transitive.*

Problem 7. *Prove that if you color the edges of K_6 in red and blue, you are guarteed to have a monochromatic triangle.*

Solution. Assume we are coloring the edges of K_6 in blue and red. Let v be a vertex of K_6. Then v has 5 edges adjacent to it. Thus at least three of these edges must be colored red. Then the picture below finishes the proof:

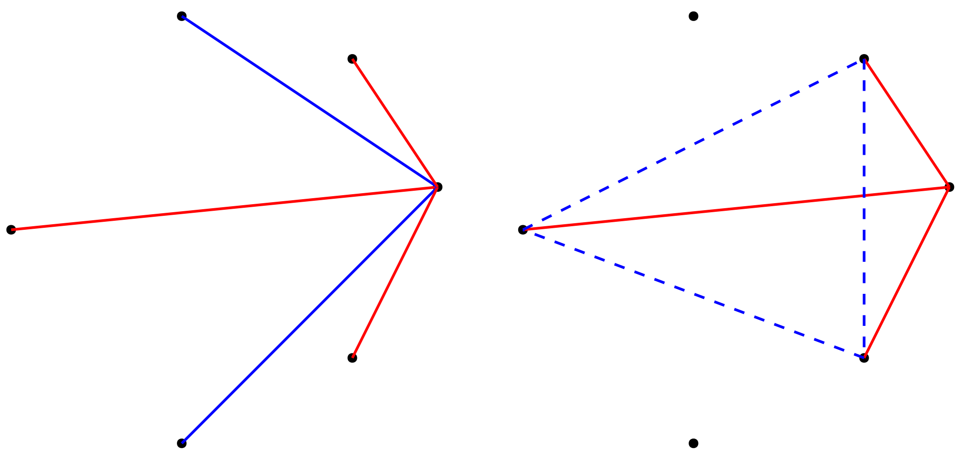

Figure 1: If any edge between two adjacent edges is red, we get a red triangle, otherwise we get a blue one.

□

Problem 8. *Let $\mathcal{R}_n$ be the set of all relations over the set $[n]$. A relation R is picked from $\mathcal{R}_n$ uniformly at random.*

1. *What is the probability that R is reflexive?*
2. *What is the probability that R is symmetric?*
3. *(HW) Are the events "R is reflexive" and "R is symmetric" independent?*
4. *(HW) Define the set $E_R := \{\{i\} \cup \{j\} \mid (i, j) \in R\}$. What is the probability that $([n], E_R)$ is a (simple undirected) graph?*

5. *(HW) Define $X : \mathcal{R}_n \to \mathbb{N}$ as follows:*

$$X(R) = \begin{cases} |R|, & \text{if } R \text{ is reflexive} \\ 0, & \text{otherwise} \end{cases}$$

What is the expected value of X?

Solution.

1. We first count the number of reflexive relations on $[n]$. A reflexive relation must contain all n elements of the form (a, a) and any of the other $n^2 - n$ elements. Thus there is 2^{n^2-n} reflexive relations. And as there is 2^{n^2} total relations, it follows that the wanted probability is 2^{-n}.

2. We count the number of symmetric relations and leave the rest for the reader. If a relation is symmetric then $(i, j) \in R \implies (j, i) \in R$. So we can have any of the n pairs (a, a) and for the rest we have $n^2/2 - n$ total options that we can choose since whenever we add (i, j) we automatically add (j, i) as well. Thus the total number is $2^n \cdot 2^{\frac{n^2-n}{2}}$.

□