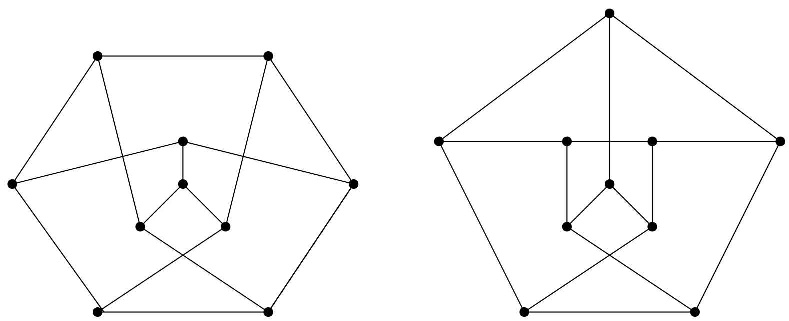
Problem 1. *Decide weather the graphs in pictures are isomorphic.*



Solution. Yes, these are 2 representations of the Petersen graph. □

Problem 2. *Decide weather sequences* $(1,1,1,2,2,3,4,4,5,5)$ *and* $(1,2,3,4,5,5,6)$ *are degree sequences of a simple graph and try to construct the graph.*

Solution. Start with the vertex of largest degree and join it with the vertices of the next largest degree. (If the largest degree is D, then we join that point with the "next" D vertices.) Repeat this procedure for the shorter degree sequence. If we arrive at the all-zero degree sequence, then the algorithm produces a simple graph with the given degree sequence. If we cannot continue without loops or multiple edges (e.g. because we get (0,0,2,0,0,...) or (0,2,2,0,...)), then there is no simple graph with the given degree sequence. □

Problem 3. *For a graph G, we say that a map $f : V(G) \to V(G)$ is an automorphism if it's a graph isomorphism. Find a nontrivial graph whose only automorphism is the identity map and prove that it is such.*

Problem 4. *Let G, H, I be three graphs and $g : G \to H$, $f : H \to I$ isomorphisms. Prove that $f \circ g : G \to I$ is an isomorphism as well.*

Solution. Let $\{u, v\} \in E(G)$, then $\{g(u), g(v)\} \in E(H)$ and $\{f(g(u)), (f(g(v))\} \in E(I)$. For the other direction, just start from $\{u, v\} \in E(I)$ and work with the inverses of f, g. □

Problem 5. *Let $\mathbb{G}$ be the set of all graphs on finite number of vertices and let $\leq$ be a relation on $\mathbb{G}$ defined by $H \leq G$ iff G contains an induced subgraph isomorphic to H. Show that $H \leq G \wedge G \leq H \implies G \simeq H$.*

Solution. If $H \leq G$, then there is an induced subgraph G' of G such that $G' \simeq H$. Now if also $G \leq H$ then there is an induced subgraph G'' of H such that $G'' \simeq G$. But since $H \simeq G'$, then G'' is also isomorphic to an induced subgraph of G and G itself, so the result follows. □

Problem 6. *Show that $\simeq$ is an equivalence relation on the set of graphs on n vertices. Show that $\simeq$ has at least $\frac{2^{\binom{n}{2}}}{n!}$ equivalence classes.*

Problem 7. *(*) Let* $G(V, E)$ *be a graph. The* excentricity *of a vertex* $V \in V$ *– denoted by* $\mathrm{ex}(v)$ *– is defined to be* $\max_{w \in V} d_G(v, w)$*, where* $d_G(x, y)$ *denotes the length of a shortest path in* G *between* x *and* y*. The* center *of a graph* G *– denoted by* $\mathrm{CT}(G)$ *– is the set* $\{v \mid \mathrm{ex}(v) \leqslant \mathrm{ex}(w) \forall w \in V\}$.

1. *Compute* $\mathrm{CT}(K_n), \mathrm{CT}(K_{m,n}), \mathrm{CT}(C_n), \mathrm{CT}(P_n)$.

2. *Show that if* G *is a tree than* $|\mathrm{CT}(G)| \leqslant 2$.

Problem 8. *(HW) For a graph* G *on* n *vertices, we define the adjacency matrix* $A(G)$ *to be the matrix which has* n *rows and* n *columns corresponding to the vertices of* G*. We define* $M(G)_{ij} = 1$ *if there is an edge between vertices* i *and* j *and* $M(G)_{ij} = 0$ *otherwise. Show that if* $M(G) = M(H)$ *then* $G \sim H$.

Problem 9. *(HW) Describe all automorphisms of* K_n *and* $K_{n,m}$ *where* $n \neq m$.