Discrete Math
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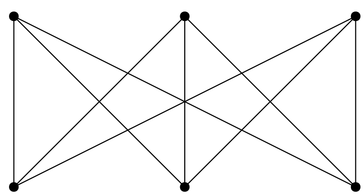
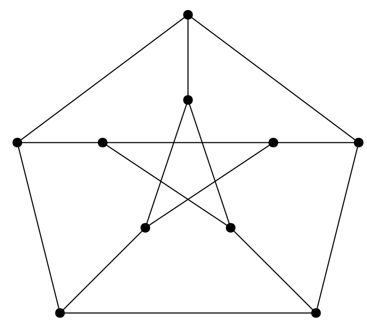
Tutorial 10
HW Due: 27.12.2023

Problem 1. *Find the chromatic number of P_n, C_n and K_n for all value of n.*

Solution. P_n is bipartite and hence has chromatic number 2 for each value of n. Similarly for cycles of even length. But what happens for C_n where n is odd? Well, we can definetly use 3 colors, since taking out 1 vertex leaves us with a path of even length which can be colored with two colors so assigning a third color to the vertex we took our doesn't cause problems. If we want to color it in two colors however, we run into a problem. Start by assigning first color to any vertex. The rest of the graph is a path of even length whose enpoints have the same color, but such a coloring of a path of even length is clearly impossible. □

Problem 2. *Find the chromatic number of the graphs in pictures.*





Solution. The graph on the left is 2-colorable since it's bipartite. Graph on the right is 3-colorable. □

Problem 3. *We say that a graph G on n vertices is k-degenerate if each* ***induced*** *subgraph H of G contains a vertex of degree at most k. Show that a graph is k-degenerate iff each subgraph contains a vertex of degree at most k.*

Solution. Let G be a k-degenerate graph and H a subgraph of G. Then, the subgraph induced by $V(H)$ has a vertex u of degree at most k. Since $u \in V(H)$, degree of u in H is also at most k. Other direction is easier. If every subgraph has a vertex of degree at most k then so does every induced subgraph since an induced subgraph is a subgraph. □

Problem 4. *Show that there is no graph G, such that G has 6 vertices and 13 edges and $\chi(G) \leq 3$.*

Solution. Any graph G with 6 vertices and 13 edges is K_6 with two edges taken out. If the two edges share a vertex, then G contains a copy of K_5 so it needs at least 5 colors. If the edges do not share a vertex, then it contains a copy of K_4. □

Problem 5. *Let G be a graph without two disjoint odd cycles. Prove that $\chi(G) \leq 5$.*

Solution. We can assume that G contains at least one odd cycle C. Then we can color C in 3 colors. Further, since every two odd cycles in G contain at least one vertex in common, we know that $G - V(C)$ contains no odd cycles and is 2-colorable. The result follows. □

Problem 6. *Show that a graph G on n vertices is k-degenerate if and only if admits a linear ordering $v_1 < v_2 < ... < v_n$ on the vertices such that each v_i forms at most k edges with vertices coming before it in the ordering.*

Solution. $\Longleftarrow$: Let G be a graph with according ordering. Let H be an induced subgraph of G. Then consider the maximal vertex in H with respect to the ordering, call it u. Then u has at most k vertices adjacent to it in H since all of them are smaller in the ordering. $\Longrightarrow$: Since every induced subgraph of G has a vertex of degree at most k, so does G. So we let this vertex be the last one in the ordering, i.e we label it v_n. Then for each $i = n-1, ..., 1$, we say that v_i is the vertex of degree at most k in $G - \{v_n, .., v_{i+1}\}$. It is easy to check that this gives the desired ordering. □

Problem 7. *(*) We say that a graph G is outerplanar if it can be drawn in the plane without edge crossings and with all vertices on the outer face, A dual graph of a planar graph G is the graph G^* whose vertices correspond to faces of G and two faces are connected by an edge if they share at least one edge.*

1. *Show that every subgraph of an outerplanar graph is outerplanar.*
2. *Prove that the dual of an outerplanar graph is a forest.*
3. *Conclude that every outerplanar graph has a vertex of degree* 2.
4. *Prove that every outerplanar graph is* 3*-colorable*

Solution. To see that a subgraph H of an outerplanar graph G is outerplanar, just consider the drawing of H inside the outerplanar drawing of G, it will clearly be outerplanar as well. To show that G^* is a tree, assume that it has a cycle. Cycle of length k in G^* corresponds to k faces $F_1, ..., F_k$ such that each F_i shares an edge with F_{i-1}, F_{i+1}. Then it is not hard to see that this forces us to have vertices on a face different than the outer face, contradicting the outerplanarity of G. Now since G^* is a tree it has a vertex of degree 1, this corresponds to a face that shares an edge with only one other face, and such a face must have a vertex of degree 2. Lastly, since G every subgraph of G is outerplanar, it also has a vertex of degree 2, so G is 2-degenerate and thus 3-colorable. □

Problem 8. *(HW) Let G be a planar, triangle-free graph. Use Euler theorem to prove that G contains a vertex of degree at most three. Then use this to prove that $\chi(G) \leq 4$. You might want to use induction.*

Solution. First we know that by Euler's formula, $v - e + f = 2$. Then, since each face is bounded by at least 4 edges, it follows that $\frac{4f}{2} < 2e \implies 2f < e \implies v > \frac{e}{2} + 2$. Now, if we assume that each vertex has degree at least 4, we obtain by a similar simple calculation that $v < \frac{e}{2}$ giving us a contradiction. Now for the second part we proceed by induction on the number of vertices in G. The result obviously holds for graphs with a single vertex. Assume that it also holds for graphs with n vertices. In this case consider a planar, triangle-free graph with $n + 1$ vertices. It has a vertex v of degree at most 3. Then consider the graph $G - v$, obtained by removing v from G. By inductive assumption it can be 4-colored. Then when we add back v, we can color it in one of the colors as it's neigbours can have at most 3 distinct colors.

□

Problem 9. *(HW) Let G be a graph on n vertices. We call an induced subgraph H of G a clique, if it is isomorphic to K_l for some value of l and we call an it an independent set if it is isomorphic to an empty graph. We denote the sizes of the largest clique and independent set of G by $\omega(G)$ and $\alpha(G)$ respectively. With this, show the following:*

1. $\chi(G) \geq \omega(G)$
2. $\chi(G) \geq \frac{n}{\alpha(G)}$

Solution.

If $\omega(G) = k$, then G contains a copy of K_k and thus needs at least k colors to be colored. The result follows.

If $\chi(G) = k$, then we can partition $V(G)$ into k independent sets $V_1, .., V_k$. Then $|V(G)| = \sum_{i=1}^{k} |V_i| \leq k\alpha(G)$ and the result follows. $\square$