Discrete Math
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Tutorial 9
HW Due: 20.12.2023

Problem 1. *Three unfriendly neighbours use the same water, beer and food sources. In order to avoid meeting, they wish to build non-crossing paths from each of their houses to each of the three sources. Can this be done?*

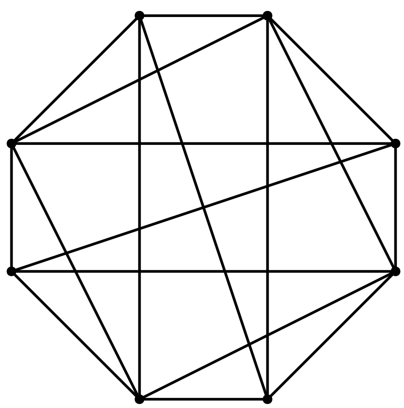
Solution. The three neigbours houses and the three sources form an instance of the graph $K_{3,3}$ where edges are formed by paths between houses and sources. Since we know that $K_{3,3}$ is not planar, construction of such paths is impossible. □

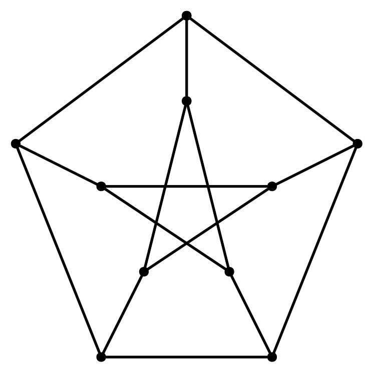
Problem 2. 1. *Draw two non-isomorphic planar graphs with the same number of vertices, edges, and faces.*

2. *Draw two planar graphs with the same number of vertices and edges, but different number of faces.*

3. *Can the graphs in the previous question be isomorphic?*

Problem 3. *Draw the below graphs with as few crossings as possible.*





Solution. Left for the reader. □

Problem 4. *Let G be a simple, connected, planar graph on n vertices and m edges. Use Euler's formula to prove that $m \leq 3n - 6$. Now assume that G also has no triangles and prove that in this case $m \leq 2n - 4$.*

Solution. Assume that we have a plane drawing of G. If we allow triangles then each face is bounded by at least 3 edges. Then if we denote the number of faces by f, it follows that $3f \leq 2m$. Factor 2 appears next to m because each edge bounds two faces. Combining this inequality with Euler's formula gives the result. In the case where we forbid triangles, the solution is the same but we replace $3f \leq 2m$ by $4f \leq 2m$. □

Problem 5. *For a graph G, we define the line graph of G, $L(G)$ to be the graph such that $V(L(G)) = E(G)$ and e, e' form an edge in $L(G)$ if $e \cap e' \neq \emptyset$. Prove that if G is connected then so is $L(G)$.*

Solution. Suppose G is connected. Then for any two vertices $u, v \in G$, there is a path $P = u \to x_1 \to x_2 \to \cdots \to x_n \to v$ where the x_i's are some other vertices of G.

By the existence of such a path P, we can say that u is adjacent to x_1, x_i is adjacent to x_{i+1} (for $1 \leq i \leq n$) and x_n is adjacent to v

By the definition of the line graph $L(G)$, this means $(u, x_1), (x_i, x_{i+1})$ and (x_n, v) are vertices of $L(G)$ [for $1 \leq i \leq n$]

Now, since (u, x_1) and (x_1, x_2) share a common endpoint x_1, they must be adjacent in $L(G)$. Similarly, (x_i, x_{i+1}) is adjacent to (x_{i+1}, x_{i+2}) for $1 \leq i \leq n-1$ and (x_{n-1}, x_n) is adjacent to $L(G)$

So, we have a path L_P in $L(G)$ as $L_p = (u, x_1) \to (x_1, x_2) \to \cdots \to (x_{n-1}, x_n) \to (x_n, v)$

Hence, for every path P in G, there is a path L_P in $L(G)$.

□

Problem 6. *(*) For any natural number n define the graphs $H_n = (V_n, E_n)$ as follows:*

$$V_n = \{0, 1, \ldots, 2^n - 1\},$$

$$E_0 = \emptyset, E_{n+1} = E_n \cup \{\{2^n + i, 2^n + j\} \mid \{i, j\} \in E_n\} \cup \{\{i, 2^n + i\} \mid 0 \leqslant i \leqslant 2^n - 1\}.$$

1. *Draw H_n for $n = 3$.*

2. *For which values of n is H_n planar.*

Problem 7. *Let $\mathcal{G}$ be a set of graphs such that for no two distinct $G, H \in \mathcal{G}$ are isomorphic to each other. Let $\preceq$ be a relation over $\mathcal{G}$ defined as follows: $H \preceq G$ iff H is a minor of G. Prove that $(\mathcal{G}, \preceq)$ is a poset.*

Solution. Reflexivity is obvious since G can be obtained from G without any contractions nor deletions. For antysimmetry suppose that H is a minor of G and G is a minor of H. Then G can be obtained from H by removing vertices and edges and vice versa. It then easily follows that G and H must be isomorphic, which is impossible in $\mathcal{G}$ unless they're the same graph. Transitivity easily follows since if H is a minor of G and G is a minor of F I can obtain H from F by first performing the minor operations needed to get G from F and then the minor operations needed to get H from G □

Problem 8. *A graph G is called* outerplanar *if it can be drawn in the plane in such a way every vertex of G lies on the outer face.*

1. *Prove that K_4 and $K_{2,3}$ are planar but not outerplanar.*

2. *Prove that every outerplanar graph contains a vertex of degree 2 or less.*

Solution. We only prove first part since the second part was shown in a later tutorial. For the first part assume for contradiction that K_4 is outerplanar. Add a vertex in the outer face and connect it to all of the other vertices. This gives us a planar embedding of K_5 which we know is impossible. Similarly for $K_{2,3}$ but we replace 5 with $3, 3$. □

Problem 9. *Prove that in each drawing of K_n for $n > 5$, there is at least $\frac{1}{5}\binom{n}{4}$ crossings. Use the non-planarity of K_5*

Solution. Write $cr(n)$ for the crossing number of K_n. Pick any 5 vertices of K_n. Then the K_5 induced by them contains at least one crossing. So that would give us at leasted $\binom{n}{5}$ crossings. But we might have counted crossings multiple times. In particular, each crossing is counted at most $n-4$ times. Thus we have $cr(n)\cdot(n-4) \geq \binom{n}{5} \implies cr(n) \geq \frac{\binom{n}{5}}{n-4} = \frac{1}{5}\binom{n}{4}$. □

Problem 10. *(HW) Prove that there is a number n_0 such that for any graph with $n \geqslant n_0$ vertices, either G or $\overline{G}$ is not planar.*

Problem 11. *(HW) Characterize all values of m, n such that $K_{m,n}$ is planar.*