
Problem 1. *Draw all non-isomorphic graphs with vertex set* $\{1, 2, 3, 4\}$.

Solution. Left for the reader. □

Problem 2. *Draw all non-isomorphic trees on* 6 *vertices.*

Solution. Left to the reader. □

Problem 3. *Prove that each graph on* n *vertices with* c *components has at least* $n - c$ *edges.*

Solution. Let $n_1, n_2, ..., n_c$ be the sizes of vertex sets of each component. Since each component is connected it has at least $n_i - 1$ edges. Hence there are at least $\sum_{i=1}^{c}(n_i - 1) = n - c$ edges in the graph! □

Problem 4. *Prove that edges of each eulerian graph can be decomposed into a disjoint union of cycles. Use induction on the number of edges!*

Solution.
Consider an Eulerian graph G and assume that the statement holds for each graph with at most as many edges as G. Take an Eulerian trail in G and fix a vertex v, then the edges between first two occurrences of v form a cycle C. Take out the edges of C from G, call this graph $G - C$. Then $G - C$ is a graph in each vertex again has an even degree. Thus by induction we can decompose $G - C$ into cycles. □

Problem 5. *Prove that a graph and it's complement can't both be disconnected.*

Solution. Suppose G is disconnected with connected components $G_1, G_2, ..., G_m$. Consider two vertices $u, v \in G_i$. In $\bar{G}$ they are connected since both will be connected to every vertex in other components of G. If u, v are in two distinct components of G then they're obviously connected in $\bar{G}$, proving the result. □

Problem 6. *Suppose that a tree has a vertex of degree* k*, show that it has at least* k *leaves.*

Solution. Let T be a tree with a vertex v of degree k. Then the graph obtained by deleting v from T is a collection of k disjoint trees each of which contains at least one leaf proving the statement. □

Problem 7. *Graph* G *is* 2*-connected if each two vertices* $u, v \in V(G)$ *are connected by two vertex-disjoint paths in* G*. Diameter of* G *is defined as* $diam(G) = \max_{u,v \in V(G)} d(u, v)$*. where* $d(u, v)$ *is the length of the shortest path between* u *and* v*. With these definitions prove the following:*

1. *Show that in a* 2*-connected graph each vertex is contained in a cycle.*
2. *Show that there is no graph* G *such that both* G *and* $\overline{G}$ *have diameter greater than three. That is, if* G *has diameter at least* 4 *then* $\overline{G}$ *has diameter at most* 3.

Solution.

1. Let x be vertex of G and consider the two vertex-disjoint paths from x to v where v is another vertex of G. Write $(x, v_0, v_1, ..., v_k, v)$ and $(x, w_0, ..., w_s, x)$ for the two paths. Then the concatenation of those two paths gives a cycle $(x, v_0, v_1, ..., v_k, v, w_s, w_{s-1}, ..., w_1, x$ which contains x!.

2. We can assume that G is connected as otherwise the statement is trivially true. We write d for distance in G and $\bar{d}$ for distance in $\bar{G}$. Take $u, v \in V(G)$ such that $d(v, u) = 3$ and let (v, x, y, u) be the shortest path from v to u in G. Let a, b be vertices in G. Since $d(u, v) = 3$, it's impossible that a is adjacent to both u, v in G. Thus without loss of generality a is adjacent to v in $\bar{G}$. Similarly we can assume that b is adjacent to u in $\bar{G}$. FInally since u, v are adjecent in $\bar{G}$ the result follows.

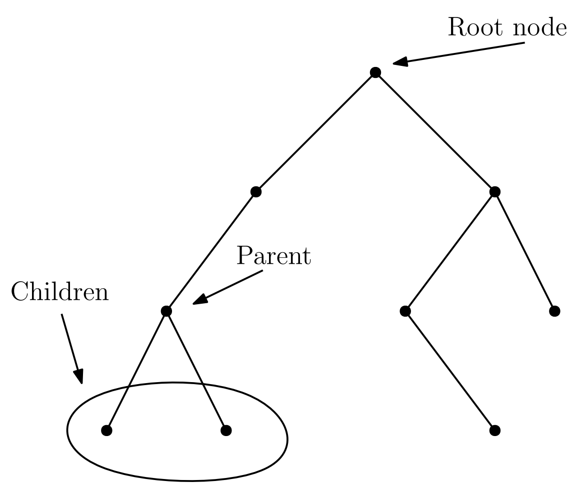
□

Problem 8. *(*) Let G be a graph on $2k$ vertices in which not three vertices form a triangle. Prove that G has at most k^2 edges using induction on k.*

Problem 9. *(HW) Prove that every k-regular graph contains P_k as a subgraph.*

Solution. We will build our path in the following way. Start from any vertex of a k-regular graph G, call it v_1. Choose any of it's k neighbours which wasn't choosen already, call it v_2. We can continue doing this until at some point all of the neighbours of the vertex we arrived to have been used. But this vertex has k neighbours, meaning we have constructed a path of length at least k. □

Problem 10. *(HW) A* rooted binary tree *is a tree where one of the vertices is labeled as* root vertex *and each node has zero, one or two child vertices connected to it (see picture below for example). Catalan numbers C_n are defined as follows: $C_0 = 1$ and $C_n = \sum_{i=0}^{n} C_{i-1}C_{n-i}$. Prove that the number of rooted binary trees on n vertices is equal to C_n.*
Hint: Consider a tree t with $n+1$ parent nodes, what can you say about the left and right subtrees of t?



Solution. Catalan numbers satisfy the recurrence:

$C_0 = 1, C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, n \geq 0$

So it suffices that show that binary trees satisfy the same recurrence.

Let T_n be the number of binary trees with n parent nodes.

There is 1 tree with zero parent nodes. So $T_0 = 1$.

For $n \geq 0$: A tree t with $n + 1$ parent nodes has a root with two subtrees as children t_1 and t_2. Since the root of t is a parent node, t_1 and t_2 must have n parent nodes together (i.e. if t_1 has i parent nodes then t_2 has $n - i$ parent nodes). Then the number of ways to make children t_1 and t_2 is $\sum_{i=0}^{n} T_i T_{n-i}$.

□