Discrete Math
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Tutorial 7
HW Due: 07.12.2023

Problem 1. *Let* (Ω, P) *be a discrete probability space and let* $X : \Omega \to \mathbb{R}$ *be a random variable. Prove that if* $\mathbb{E}[X] < a$ *for some* $a \in \mathbb{R}$*, then there exists an outcome* $\omega \in \Omega$ *such that* $X(\omega) < a$*.*

Solution. Assume that for all $\omega \in \Omega$, we have $X(\omega) \geq a$. Then $\mathbb{E}[X] = \sum_{\omega\in\Omega} \mathbb{P}(\omega)X(\omega) \geq a \cdot \sum_{\omega\in\Omega} \mathbb{P}(\omega) = a$. □

Problem 2. *Let* $G = (V, E)$ *be a graph. The* degree *of a vertex* $v \in V$ *– denoted by* $\delta_G(v)$ *– is the number of edges incident to* v*. In other words,*

$$\delta_G(v) := |\{e \in E \ \mid \ v \in e\}|.$$

The average degree of a graph G *is defined to be the average of the degrees of the vertices, that is,* $\dfrac{\sum_{v\in V(G)} \delta_G(v)}{|V(G)|}$*. Compute the average degrees of* $K_n, K_{m,n}, P_n,$ *and* C_n*.*

Solution.

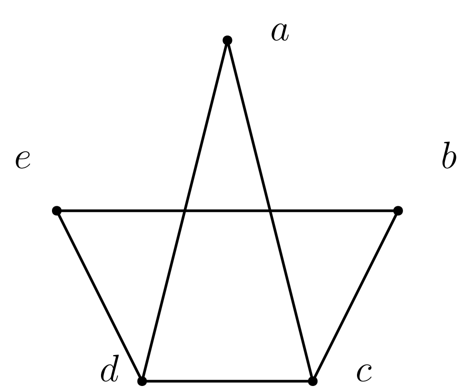
For K_n, each vertex has degree $n-1$ and hence that is the average degree. For $K_{m,n}$ there is m vertices with degree n and n vertices with degree m. Hence the average degree is $\frac{2mn}{m+n}$. In P_n there is $n-2$ vertices of degree 2 and 2 vertices of degree 1 hence the average degree is $\frac{2n-2}{n}$. In C_n each vertex has degree 2 and that's the average degree as well.

□

Problem 3. *Among a group of 5 people, is it possible for everyone to be friends with exactly 2 of the people in the group? What about 3 of the people in the group?*

Solution. It is possible for everyone to be friends with exactly 2 people. You could arrange the 5 people in a circle and say that everyone is friends with the two people on either side of them (so you get the graph C_5). However, it is not possible for everyone to be friends with 3 people. That would lead to a graph with an odd number of odd degree vertices which is impossible since the sum of the degrees must be even. □

Problem 4. *Is the graph* $G = (V, E)$ *where* $V = \{a, b, c, d, e\}$*, and* $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$ *equal to the graph* H *drawn below? Are they isomorphic?*



Solution. The graphs are not equal. For example, graph G has an edge $\{a, b\}$ but graph H does not have that edge. They are isomorphic. One possible isomorphism is $f : V(G) \to V(H)$ defined by $f(a) = d, f(b) = c, f(c) = e, f(d) = b, f(e) = a$. □

Problem 5. *Suppose a subset of* $[n]$ *is picked uniformly at random. Consider the random variable* $X : 2^{[n]} \to \mathbb{R}$ *with* $X(S) = \max S$. *For example,* $\max\{3, 2, 6, 4\} = 6$. *Compute the expected value of* X

Problem 6. *Show that there is a way to color the edges of* K_n, $n > 5$ *with two colors in such a way that there is at most* $\binom{n}{5}2^{1-\binom{5}{2}}$ *monochromatic copies of* K_5. *(Hint: Use linearity of expectation!)*

Solution. We color each edge in K_n red or blue at random. Let $X \subseteq V(K_n), |X| = 5$ and let I_X be the indicator variable for the event that the graph induced by X is monochromatic. Then $P(I_X = 1) = 2^{1-\binom{5}{2}}$. This is because we can choose one edge of the graph induced by X and then X is monochromatic if and only if all of the other edges are colored by the same color. Then, as there are $\binom{n}{5}$ subsets of $V(K_n)$ of size 5 the result follows by linearity of expectation. □

Show that there is a way to color the edges of $K_{n,m}$, $n, m > 5$ with two colors in such a way that there is at most $\binom{n}{5}\binom{m}{5}2^{1-25}$ monochromatic copies of $K_{5,5}$. Use linearity of expectation!

Problem 7. *(*) Let* G *be a graph on* n *vertices. Let* d *be it's average degree,* Δ *the maximal degree and* α *the size of the largest independent set (induced subgraph with no edges). Then prove the following:*

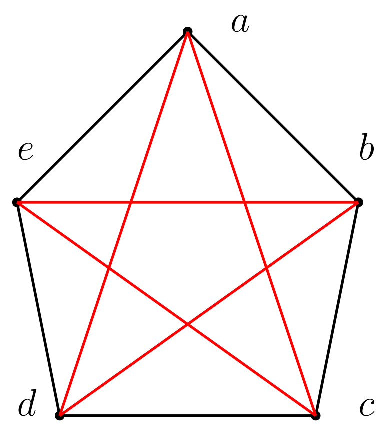
1. $\alpha(G) \geq \frac{n}{\Delta+1}$
2. $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{deg(v)+1}$

Problem 8. *(HW) The complement of a graph* $G = (V, E)$ *is the graph* $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$. *Show that two graphs are isomorphic if and only if their complements are isomorphic.*

Solution. Let G, H be graphs and $f : G \to H$ an isomorphism. Then $\{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H)$. Or equivalently $\{u, v\} \in E(\bar{G}) \iff \{u, v\} \notin E(G) \iff \{f(u), f(v)\} \notin E(H) \iff \{u, v\} \in E(\bar{H})$. □

Problem 9. *(HW) We call a graph* G *self-complementary if it is isomorphic to its complement* $\overline{G}$. *Find all self-complementary cycles and prove that no others exist.*

Solution. For a cycle graph to be self-complementary, the complement graph must have the same number of edges as its original. This can only happen when $n = 5$, as the number of edges in $G + \bar{G} = \frac{n(n-1)}{2}$, so we require $n(n-1) = 4n$, or $n^2 - 5n = 0$ which only has solutions $n = 0, 5$. To check that K_5 is self complementary, consider the picture below, the C_5 is drawn in black and has cycle structure (a, b, c, d, e) while it's complement has cycle structure (a, c, e, b, d).


a
e
b
d
c

□