Problem 1. *List all graphs with vertex set* $\{1, 2, 3, 4\}$ *and two edges.*

Problem 2. *Let* X_n *be the sum of numbers obtained by rolling the dice* n *times, compute* $\mathbb{E}(X_n)$.

Solution. Let X be the random variable corresponding to throwing the dice once. Clearly $\mathbb{E}(X) = 1/6 + 2/6 + 3/6 + \cdots + 6/6 = 3.5$. Then we can notice that $X_n = n \cdot X$ and hence by linearity of expectation $\mathbb{E}(X_n) = 3.5n$. □

Problem 3. *(Bayes' Theorem) Let* (Ω, P) *be a finite probability space,* A *an event, and* $B_1, \ldots, B_n$ *a partition of* Ω.

1. *Prove that* $P[A] = \sum_{i=1}^{n} P[A \cap B_i]$.
2. *Let* $i \in \{1, \ldots, n\}$. *Prove that* $P[B_i|A] = \frac{P[A|B_i] \cdot P[B_i]}{P[A]}$.
3. *Let* $i \in \{1, \ldots, n\}$. *Prove that*

$$P[B_i|A] = \frac{P[A|B_i] \cdot P[B_i]}{\sum_{j=1}^{n} P[A|B_j] \cdot P[B_j]}.$$

Solution.

1. Since $B_1, \ldots, B_n$ is a partition of Ω, $A \cap B_1, \ldots, A \cap B_n$ is a partition of A and the result follows since $(A \cap B_i) \cap (A \cap B_j) = \emptyset$ for $i \neq j$.
2. $P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i)P(B_i)}{P(A)}$, where the last equality follows from definition of of conditional probability.
3. We only need to prove that $P(A) = \sum_{j=1}^{n} P(A|B_j)P(B_j)$. But this follows immediately from Part 1. and the definition of conditional probability.

□

Problem 4. *A box contains three coins: two regular coins and one fake two-headed coin* $(P(H) = 1)$,

1. *You pick a coin at random and toss it. What is the probability that it lands heads up?*
2. *You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?*

Solution.

1. We can use the fact that $P(A) = \sum_{j=1}^{n} P(A|B_j)P(B_j)$ with $n = 2$ and B_1 corresponding to the event that the chosen coin is fair and B_2 corresponding to the event that it isn't. If A is the event that coin shows head after the flip, then $P(A) = 1/2 \cdot 2/3 + 1 \cdot 1/3$.

2. We use the same definition of n, B_1, B_2, A as before. Then by application of part 2 of problem 3 we get the result:

$$P(B_2|A) = \frac{1/3}{2/3} = 1/2$$

□

Problem 5. *Is it possible to have a tournament where for every subset S of k players, there exists another player who has defeated every player in S?*

Let T be a random tournament with n players $P = \{p_1, \ldots, p_k\}$ where for every pair p, q of players the winner is decided by the toss of a fair coin.

1. *Let $S = \{p_1, \ldots, p_k\}$, and let $w \notin S$. What is the probability that player w is defeated by some player in S?*

2. *Let $S = \{p_1, \ldots, p_k\}$, and let $w \notin S$. What is the probability that for every player $w \notin S$, w is defeated by some player in S?*

3. ** Prove that the probability that a random tournament does not meet the requirement stated at the beginning of the problem is strictly less than 1 for large enough n.*

Solution.

1. We first consider the probability that w defeats every player in S. Probability that w defeats a player from S is $1/2$ and since each match is independent probability that w defeats every player in S is 2^{-k}. Then the probability that w is defeated by at least one player in S is $1 - 2^k$.

2. Write $w_1, ..., w_{n-k}$ be the players not in S. Further, the events that W_i and w_j are defeated by a player from S are independent for $i \neq j$, so the desired probability is $(1 - 2^k)^{n-k}$.

3. There is $\binom{n}{k}$ choices of S. Let I_S be the event that every player outside S was defeated by a player in S. Then $\mathbb{E}(I_S) = (1 - 2^k)^{n-k}$. Then by linearity of expectation the expected number of groups beaten by no player is $\binom{n}{k}(1 - 2^k)^{n-k}$. The desired tournament exists when this number is less than 1.

□

Problem 6. *Seat b black and r red knights at random around a round table on chairs numbered counterclockwise from 1 to $n = b + r$ so that all seating arrangement are equaly likely. Let X be the number of black knights that have a black knight to their right. Compute $\mathbb{E}(X)$.*

Solution. Define an indicator I_k which is 1 if seats k and $k+1$ have black nights and 0 otherwise. Then $E(I_k) = \frac{\binom{b}{2}}{\binom{n}{2}}$. Then we can express X as the sum of I_k's for all possible values of k. Finally, the solution is obtained by linearity of expectation! □

Problem 7. ** In an $n \times n$ array, each of the numbers $1, 2, \ldots, n$ appears exactly n times. Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct numbers.*

Problem 8. ** A fair coin is tossed until two consecutive tails appear. The tosses are independent. Denote by N the number of necessary tosses including the last two tails. Express the probabilities $P(n = N)$ for $n \geq 2$ in terms of Fibonacci sequence $\{F_n\}_{n\geq 1}$ given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.* ***Hint:*** *condition on first two tosses and derive a recursive formula for the sequence $a_n = 2^n P(N = n)$.*

Problem 9. *Prove that there is a constant c such that whenever we have two natural numbers n, m such that $n > cm^2$ then a random mapping $f : \{1, 2, ..., n\} \to \{1, 2, ..., m\}$ is surjective with probability at least 0.99. IMPORTANT: You will need to use the fact that for any $p \in (0, 1)$, $1 - p \leq e^{-p}$.*

Solution. For each $k \in \{1, 2, ..., m\}$ define an event $X_k = \{k$ is not in the image of f. Then $P(X_k) = \frac{(m-1)^n}{m^n} = 1 - \frac{1}{m}$. Then we can do the following calculation.

$$P(\text{f is not surjective}) = P(\bigcup_k X_k) \leq \sum_k P(X_k) =$$

$$= m(1 - \frac{1}{m})^n \leq me^{-\frac{n}{m}}$$

. Now we want $e^{-\frac{n}{m}} \leq e^{-cm<0.01}$. Simple logartithm calculation then gives that this works for $c > ln(100)$. □

Problem 10. *(HW) Let (Ω, P) be a finite probability space, and let $A_1, A_2, \ldots, A_n$ be events. Prove that*

$$P[\bigcup_{i=1}^{n} A_i] = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} P[\bigcap_{i \in I} A_i].$$

Problem 11. *(HW) Prove that there is a constant c such that whenever we have two natural numbers n, m such that $m > cn^2$ then a random mapping $f : \{1, 2, ..., n\} \to \{1, 2, ..., m\}$ is injective with probability at least 0.99.*