Discrete Math — Tutorial 5

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Problem 1. *Assume that we are flipping a fair coin 6 times. What is the probability of the event "There is an even number or heads or there is exactly 3 heads and 3 tails"?*

Solution. The set of all possible outcomes has 2^6 elements since each flip can result in heads or tails. There is $\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6}$ possible outcomes in which the number of heads is even. How to get this? Where let's say we want to count the number of ways that flipping the coins 6 times gives us i heads. This is equal to choosing which i of the 6 flips give you head or $\binom{6}{i}$. Then we need to sum over all possible even choices of i. By similar calculation, the number of possible outcomes in which we have 3 heads and 3 tails is $\binom{6}{3}$, so the final solution is

$$P(\text{even num of heads or 3 heads}) = \frac{\binom{6}{0}+\binom{6}{2}+\binom{6}{4}+\binom{6}{6}+\binom{6}{3}}{2^6}.$$

□

Problem 2. *A standard deck of 52 cards is dealt out to 4 players so that each player gets 4 cards. What is the probability that none of the 4 players have cards of all four suits? You do not need to simplify the binomial symbols.*

Solution. Denote by A the event that none of the 4 players have cards of all four suits. Further denote by B_k the event that the k-th player has cards of all four suits for $k = 1, 2, 3, 4$. Then $P(A) = 1 - P(B_1 \cup B_2 \cup B_3 \cup B_4)$. To calculate probability of B_i we need to consider all possible ways to deal out 4 cards to the player which is $\binom{52}{4}$. Further we need to consider the cases in which he received one card from each suit. As there are 13 cards of each suit and we need to pick one of each there are 13^4 ways to do this. Finally $P(B_i) = \frac{13^4}{\binom{52}{4}}$. If we want to calculate $P(B_i \cap B_j)$ we need to choose 4 cards for one and another 4 cards for the second player for $\binom{52}{4}\binom{48}{4}$ possible outcomes. Then we choose one card from each suit for the first player and one card from each suit for the second. This gives us $13^4 * 12^4$ ways to do this and $P(B_i \cap B_j) = \frac{13^4 * 12^4}{\binom{52}{4}\binom{48}{4}}$. Similar computation gives $P(B_i \cap B_j \cap B_k) = \frac{13^4 12^4 11^4}{\binom{52}{4}\binom{48}{4}\binom{44}{4}}$ and $P(\bigcap_{i=1}^4 B_i) = \frac{13^4 12^4 11^4 10^4}{\binom{52}{4}\binom{48}{4}\binom{44}{4}\binom{40}{4}}$. The rest of the solution is computation via inclusion-exclusion formula. □

Problem 3. *Suppose $m > 1$ begonias and $n > 1$ fuchsias are randomly arranged on a window sill. All orderings of the $m + n$ flowers are equally likely. What is the probability that to the right of the leftmost begonia there is another begonia?*

Solution. We can count the number of arrangments in which two leftmost begonias are next to each other by considering them as a single flower and placing them together. Thus there are $\binom{m+n-1}{m-1}$ such arrangments. As there is a total of $\binom{n+m}{m}$ total arrangments of begonias and fuchasias one can compute the probability easily. □

Problem 4. *A nonnegative integer solution of $x_1 + x_2 + x_3 = 11$ is picked uniformly at random. What is the probability that $x_1 \leqslant 3, x_2 \leqslant 4,$ and $x_3 \leqslant 6$ in the chosen solution?*

Solution. We first find the number of possible outcomes which is equal to the number of nonegative solutions to the equation. This number is $\binom{11+3-1}{3-1}$. Now we need to find the number of possible solutions to the equation that satisfy the constraints above. To do this just follow the procedure from the previous tutorial sheet. Lastly to find the probability that the solution satisfies the constraints just divide the two numbers. □

Problem 5. *Let (Ω, P) be a probability space and B an event. Consider a function $P' : 2^B \to [0,1]$ defined as $P'(X) = P(X)/P(B)$. Prove that (B, P') forms a probability space.*

Solution. Our sample space is B, events are all subsets of B. And the σ-algebra properties of the space are directly inherited from Ω. Thus, the only things that needs proving is that P' defines a proper probability function. Obviously $P'(B) = P(B)/P(B) = 1$. To check subadditivity let $A_1, A_2, ...$ be a countable collection of subsets of B. THen $P(\sum_i = 1^\infty P(A_i) = \frac{\sum_i = 1^\infty P(A_i)}{P(B)} \leq \frac{P(\bigcup_{i=1}^\infty A_i)}{P(B)} = P'(\bigcup_{i=1}^\infty A_i)$, finishing the proof. □

Problem 6. *(*) A set of events $A_1, ..., A_i$ is said to be independent if $P(A_1 \cap A_2 \cap \cdots \cap A_i) = \prod_{j=1}^i P(A_j)$. Now take a probability space with 8 elements where each event, i.e., equally likely and in it 4 events A, B, C, D such that each triple of events is independent but all* 4 *events aren't.*

Problem 7. *Let π be a permutation of the set $\{1, \ldots, n\}$. For $1 \leqslant i \leqslant n$, we call i a* fixed point *of π if $\pi(i) = i$. A permutation is called a* derangement *if it has no fixed points.*

1. *List all derangements of $\{1, 2, 3, 4\}$.*
2. *Compute the number of derangements of $\{1, \ldots, n\}$.*
3. *What is the probability that a permutation of $\{1, \ldots, n\}$, picked uniformly at random, is a derangement as $n \to \infty$?*

Solution. We will leave part one to the student and only give solutions to part two and three. To count the number of dearangments of $\{1, 2, ..., n\}$ we use the inclusion-exclusion principle. Let A_i be the set of all permutations fixing the number i. We have covered the calculations needed for sizes of $A_i's$ and intersection of multiple A_i's in the previous homework so we only give two examples. $|A_1| = (n-1)!$ and $A_1 \cap \cdots \cap A_i = (n-i)!$. As there is $\binom{n}{k}$ k-tuples of these sets we can calculate the formula as

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = \sum_{i=0}^{n} \frac{(-1)^i n!(n-i)!}{(n-i)!i!} = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Now using this we can compute the probability that a randomly chosen permutation is a dearangment. But we need the following fact:

$$\sum_{x=0}^{\infty} \frac{x^i}{i!} = e^x.$$

Substituting $x = (-1)$ and taking the limit we can see that the wanted probability is $1/e$! If this makes no sense don't worry, it will make at least slightly more sense after you take some analysis and combinatorics classes :). □

Problem 8. *(HW) Let R be a relation over a set X. Consider the relation $\preceq_R$ defined over $X \times X$ as follows: $(a_1, b_1) \preceq_R (a_2, b_2)$ if either $a_1 \neq a_2 \wedge (a_1, a_2) \in R$ or $a_1 = a_2 \wedge (b_1, b_2) \in R$. Prove that $\preceq_R$ is an order over $X \times X$ if and only if R is an order over X.*

Solution. Suppose, R is an ordering.

1. Since R is an ordering, it is true that $\forall b \in X$, since $(b, b) \in R$, $(a, b) \preceq_R (a, b)$ which is precisely a reflexivity.

2. Let $(a_1, b_1) \preceq_R (a_2, b_2)$ and $(a_2, b_2) \preceq_R (a_1, b_1)$ and $(a_1, b_1) \neq (a_2, b_2)$.

If $a_1 \neq a_2$, $(b_1, b_2) \in R$ and $(b_2, b_1) \in R$, which contradicts the claim that R is an ordering. Hence, $(a_1, b_1) = (a_2, b_2)$, and this implies that $\preceq_R$ is antisymmetric.

If $a_1 = a_2$, then $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$, which also contradicts the antisymmetry of R. Hence, $(a_1, b_1) = (a_2, b_2)$, and $\preceq_R$ is antisymmetric.

3. Let $(a_1, b_1) \preceq_R (a_2, b_2), (a_2, b_2) \preceq_R (a_3, b_3)$, but not $(a_1, b_1) \preceq_R (a_3, b_3)$.

Case 1. $a_1 = a_3$. Then, $(b_1, b_3) \notin R$. Since R is a transitive relation, $(b_1, b_2) \in R$ and $(b_2, b_3) \in R$ cannot be true simultaneously. Hence, either $(b_1, b_2) \notin R$ or $(b_2, b_3) \notin R$. If $(b_1, b_2) \notin R$, then $(a_1, a_2) \in R$, and hence, $a_1 \neq a_2$. If $(b_2, b_3) \notin R$, then $(a_2, a_3) \in R$, and hence, $a_2 \neq a_3$. Since $a_1 = a_3$, in either case $a_2 \neq a_1$ and $a_2 \neq a_3$. Then $(a_1, a_2) \in R$, and, consequently, $(a_3, a_2) \in R$. Since $(a_2, b_2) \preceq_R (a_3, b_3)$, $(a_2, a_3) \in R$, which contradicts that R is an ordering. Hence, $(a_1, b_1) \preceq_R (a_3, b_3)$, and $\preceq_R$ is transitive. Hence, $\preceq_R$ is an ordering.

Case 2. $a_1 \neq a_3$. This implies that $(a_1, a_3) \notin R$. Since R is a transitive relation, $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ cannot be true simultaneously. Hence, either $(a_1, a_2) \notin R$ or $(a_2, a_3) \notin R$.

If $(a_1, a_2) \notin R$, then $(b_1, b_2) \in R$, which implies that $a_1 = a_2$. Then, our assumption $(a_1, a_2) \notin R$ contradicts the reflexivity of R. Hence, $(a_1, b_1) \preceq_R (a_3, b_3)$, and $\preceq_R$ is an ordering.

If $(a_2, a_3) \notin R$, then $(b_2, b_3) \in R$, which implies that $a_2 = a_3$. Then, our assumption $(a_2, a_3) \notin R$ contradicts the reflexivity of R. Hence, $(a_1, b_1) \preceq_R (a_3, b_3)$, and $\preceq_R$ is an ordering.

Hence, R is an ordering $\Rightarrow \preceq_R$ is an ordering.

Suppose, $\preceq_R$ is an ordering.

1. Let $a, b \in X$ s.t. $(b, b) \notin R$. This implies that $(a, b) \not\preceq_R (a, b)$, which contradicts the reflexivity of $\preceq_R$. Hence, R is reflexive.

2. Let $b_1, b_2 \in X$ s.t. $(b_1, b_2) \in R$ and $(b_2, b_1) \in R$. Fix an arbitrary element $a \in X$. Then, $(a, b_1) \preceq_R (a, b_2)$ and $(a, b_2) \preceq_R (a, b_1)$, which by antisymmetry of $\preceq_R$ implies that $(a, b_1) = (a, b_2)$, which implies that $b_1 = b_2$. Hence, R is antisymmetric.

3. Let $b_1, b_2, b_3 \in X$, s.t. $(b_1, b_2) \in R$ and $(b_2, b_3) \in R$. Fix an arbitrary element $a \in X$. Since $(b_1, b_2) \in R$, $(a, b_1) \preceq_R (a, b_2)$, and since $(b_2, b_3) \in R$, $(a, b_2) \preceq_R (a, b_3)$. Since $\preceq_R$ is a transitive relation, $(a, b_1) \preceq_R (a, b_3)$, which implies that $(b_1, b_3) \in R$. Hence, R is transitive.

Hence, $\preceq_R$ is an ordering iff R is an ordering.

□

Problem 9. *(HW) Let X be a set and let R be a relation over X that is both reflexive and transitive. Define the relation $\sim_R$ over X as follows: $a \sim_R b$ iff $(a, b) \in R \wedge (b, a) \in R$.*

1. *Prove that $\sim_R$ is an equivalence relation over X.*

2. *Let $\mathcal{X}_R$ be the set of equivalence classes of $\sim_R$. Define the relation $\preceq_R$ over $\mathcal{X}_R$ as follows: $A \preceq_R B$ iff $\exists a \in A, b \in B : (a, b) \in R$. Prove that $\preceq_R$ defines an order over $\mathcal{X}_R$.*

Solution. To prove the first statement, we must prove three properties for $\sim_R$: reflexivity, symmetry and transitivity. - **Reflexivity.** Consider an arbitrary $x \in X$. R is reflexive, so $(x, x) \in R$, from which we obtain that $x \sim_R x$. Since this is true for all $x \in X$, we see that $\sim_R$ is reflexive. - **Symmetry.** Consider some $a, b \in X$ such that $a \sim_R b$. Then, by definition, $(a, b) \in R \wedge (b, a) \in R$, which is equivalent to $(b, a) \in R \wedge (a, b) \in R$. By definition of $\sim_R$, this means that $b \sim_R a$. But since this is always the case given that $a \sim_R b$, we see that $\sim_R$ is symmetric. - **Transitivity.** Consider some $a, b, c \in X$ such that $a \sim_R b$ and $b \sim_R c$. Then, from the first assumption we'll have that $(a, b) \in R$ and $(b, a) \in R$ – and from the second one we'll have that $(b, c) \in R$ and $(c, b) \in R$. Since we know that $(a, b) \in R$ and $(b, c) \in R$, by transitivity of R, this means that $(a, c) \in R$. And since we also know that $(c, b) \in R$ and $(b, a) \in R$, also by the transitivity of R we see that $(c, a) \in R$. Therefore, by definition of $\sim_R$, we see that $a \sim_R c$. Given that this is always the case if $a \sim_R b$ and $b \sim_R c$, we see that $\sim_R$ is transitive.

Given that $\sim_R$ is reflexive, symmetric and transitive, we conclude that $\sim_R$ is an equivalence relation over X.

To prove the second statement, we must prove three properties for $\preceq_R$: reflexivity, antisymmetry and transitivity. - **Reflexivity.** Consider an arbitrary $A \in \mathcal{X}_R$. Then, A is an equivalence class of $\sim_R$, which we know must be non-empty. Now, consider some $a \in A$, which also implies that $a \in X$. Therefore, by reflexivity of R, it will be the case that $(a, a) \in R$. And since there exists an $a \in A$ such that $(a, a) \in R$, we conclude that $A \preceq_R A$. Since this is the case for any $A \in \mathcal{X}_R$, we see that $\preceq_R$ is reflexive. - **Antisymmetry.** Consider some $A, B \in \mathcal{X}_R$ such that $A \preceq_R B$ and $B \preceq_R A$. Then, the first statement implies that there exist $a \in A$, $b \in B$ such that $(a, b) \in R$, and the second one implies that there exist $b' \in B$, $a' \in A$ such that $(b', a') \in R$. Now, recalling that A and B are equivalence classes on $\sim_R$, given that $a, a' \in A$, we see that $a \sim_R a'$,and given that $b, b' \in B$, we see that $b \sim_R b'$. These two statements mean that $(a, a') \in R$, $(a', a) \in R$, $(b, b') \in R$ and $(b', b) \in R$. Now, given that $(b, b') \in R$ and $(b', a') \in R$, by transitivity of R, we see that $(b, a') \in R$. But we also know that $(a', a) \in R$, so again by transitivity, $(b, a) \in R$. From the start we knew that $(a, b) \in R$, so we conclude that $a \sim_R b$, from which we know that $\sim_R [a] =\sim_R [b]$. And it is already known that $A =\sim_R [a]$ and $B =\sim_R [b]$, so we see that $A = B$. Since this is always the case given that $A \preceq_R B$ and $B \preceq_R A$, we see that $\preceq_R$ is antisymmetric. - **Transitivity.** Consider some $A, B, C \in \mathcal{X}_R$ such that $A \preceq_R B$ and $B \preceq_R C$. Then, the two statements imply that there exist $a \in A, b \in B$ such that $(a, b) \in R$, and that there exist $b' \in B, c \in C$ such that $(b', c) \in R$. Since $b, b' \in B$, with B being an equivalence class of $\sim_R$, we see that $b \sim_R b'$, from which we know that $(b, b') \in R$ and $(b', b) \in R$. Now, given that $(a, b) \in R$ and $(b, b') \in R$, by the transitivity of R, we see that $(a, b') \in R$. But furthermore, we know that $(b', c) \in R$, so again by the transitivity of R, $(a, c) \in R$. From this, knowing that $a \in A$ and that $c \in C$, we conclude that $A \preceq_R C$. And since this is always the case given that $A \preceq_R B$ and $B \preceq_R C$, we see that $\preceq_R$ is transitive.

Given that $\preceq_R$ is reflexive, antisymmetric and transitive, we conclude that $\preceq_R$ is an order over $\mathcal{X}_R$. This concludes our proof.

□