Discrete Math | Tutorial 4
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Problem 1. *Give an example of a partial order on X where X has 15 elements such that width of your partial order is 3 and length is 5.*

Solution. There is a lot of solutions but we give one example. Take our underlying set to be $\{2, 4, 8, 16, 32, 3, 9, 27, 81, 243, 5, 25, 125, 625, 3125\}$. And as for our ordering let's say that $x \leq y$ if x divides y. This is clearly a partial ordering and our longest chain is 5, for example $2 \leq 4 \leq 8 \leq 16 \leq 32$. And to see that our biggest antichain has size 3 take for example $2, 3, 5$, this is clearly an antichain. To see that it is indeed the biggest one, we can argue by contradiction and this is left as an exercise for the reader. $\square$

Problem 2. *Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?*

Solution. We can apply the inclusion exclusion principle here. Let A be the set of all bruised apples and B the set of all apples with worms. Then $|A \cup B| = 20 + 15 - 10 = 25$. So we can in fact sell $100 - 25 = 75$ apples. $\square$

Problem 3. *Describe all relations on a set X that define an equivalence relation and a partial ordering at the same time.*

Solution. We want to find all relations that are reflexive, transitive, symmetric and antisymmetric at the same time. So one obvious candidate for such a relation is $\Delta = \{(x, x) : x \in X\}$. Can we have something more? Well, assume that we have a relation R that is an equivalence and a partial order and that $(a, b) \in R$ where $a \neq b$. Then because R is symmetric we need to have $(b, a) \in R$. However R is also antisymmetric so this is impossible. Thus Δ is the only such relation that can exist. $\square$

Problem 4. *Answer/prove the following:*

1. *For a given totally ordered set $(X, <)$, any subset $A \subseteq X$ has at most one supremum and at most one infimum.*

2. *Which element is the supremum/infimum of the empty set.*

3. *Give an example of a poset in which every nonempty subset has an infimum but there are subsets with no supremum.*

Solution. We will prove only one part of the first question as the other is very similar.

1. Assume that $A \subseteq X$ has two suprema. Then because M is a supremum, every other upper bound of A bigger than it, so in particular M' is bigger than M. In a completely symmetric fashion one can see that M is bigger than M'. But since ordering is antisymmetric, we get that $M = M'$.

2. Consider the empty set as a subset of some totally ordered set X. Then for any element $x \in X$ the following holds:

$$\forall a \in \emptyset : x < a < x.$$

Why is this? Well there is no elements in the empty set so clearly all elements of the empty set are smaller and bigger than x since there is none (yes this takes some time to get into your head). Therefore the smallest element of X is the supremum of the $\emptyset$ and the biggest element of X is the infimum of the $\emptyset$.

□

Problem 5. *Let R be a relation over a set X, and let $Y \subseteq X$.*

1. *Prove that if R is an ordering over X then $R \cap (Y \times Y)$ is an ordering over Y.*
2. *Prove that if R is an equivalence relation over X then $R \cap (Y \times Y)$ is an equivalence relation over Y.*

Solution. We only prove the first part since the other is basically the same. In order to prove that $S = R \cap (Y \times Y)$ is an ordering we need to prove that S is reflexive,antisymmetric and transitive. Let $y \in Y$ we need to show that $(y, y) \in S$. But $(y, y) \in R$ since R is an ordering and $(y, y) \in Y \times Y$ obviously, hence S is reflexive. To show that S is antisymmetric, assume that $(a, b) \in S$, then $(a, b) \in R$ and since R is antisymmetric $(b, a) \notin R$. Now $(b, a) \notin S$ since $S \subseteq R$ so we get what we want. Lastly we need to prove that S is transitive. Let $a, b, c \in Y$ be such that $(a, b), (b, c) \in S \subseteq R$. Then it follows that $(a, c) \in R$ and $(a, c) \in Y \times Y$, therefore $(a, c) \in S$. □

Problem 6. *Use inclusion-exclusion principle to determine the number of integer solutions of the equation*

$$x_1 + x_2 + x_3 + x_4 = 20$$

satisfying the constraints $1 \leq x_1 \leq 6; 0 \leq x_2 \leq 7; 4 \leq x_3 \leq 8; 2 \leq x_4 \leq 6$.

Solution. We start by making a slight modification to our equation. In particular if $k \leq x_i \leq l$ we replace x_i with $y_i = x_i - k$ and we change our bounds to $0 \leq y_i \leq l - k$. This leaves us with:

$$y_1 + y_2 + y_3 + y_4 = 13$$

satisfying constraints $0 \leq y_1 \leq 5; 0 \leq y_2 \leq 7; 0 \leq y_3 \leq 4; 0 \leq y_4 \leq 4$.

So the number of non-negative integer solutions to this equation is $\binom{13+4-1}{13}$. It is important to note that these equations obey no other contsraints except that each variable is non-negative! We want to proceed by inclusion-exclusion. To start we make 4 sets A_1, A_2, A_3, A_4, where A_i is the set of all solutions of our equation where y_i is bigger than the corresponding upper bound. We will show how to find $|A_1|$ and leave the rest as practice. So solutions in A_1, are such that $y_1 \geq 6$. Thus we introduce a new variable $z_1 = y_1 - 6$. Then we have the equation

$$z_1 + y_2 + y_3 + y_4 = 7$$

Now, we find the number of nonegative solutions to the above equation as before. Therefore $|A_1| = \binom{7+4-1}{7}$.

Once you have preformed all of the calculations needed you find the number of solutions satisfying the original constraints by substracting the $|A_1 \cup A_2 \cup A_3 \cup A_4|$ which you can obtain using inclusion-exclusion. $\square$

Problem 7. *(*) Prove Erdős-Szekeres theorem*[1] *using the Pigeonhole principle.*

Problem 8. *Use Erdős-Szekeres theorem to prove the following statement: In a set P of at least $rs - r - s + 2$ points in the plane there is a polygonal path of either $r - 1$ positive slope edges or $s - 1$ negative slope edges. You can assume that no two points have same x coordinate.*

Solution. Construct a sequence $\{a_i\}$ of length $rs - r - s + 2$ the following way. Start "scanning" the plane from left to right and record the y coordinate of the i^{th} point we see as the value of a_i. Then we apply Erdős-Szekeres theorem to obtain an increasing subsequence of length $r - 1$ or a decreasing one of length $s - 1$. These correspond exactly to our wanted polygonal paths! We just need to connect the corresponding points. $\square$

Problem 9. *Use inclusion-exclusion principle to calculate the number of onto functions from X to Y where $|X| = n$ and $|Y| = m$.*

Solution. We will provide first part of the calculation and the sets that need to be used for Inclusion-exclusion and will leave the rest for an easy practice. Name the elements of Y $y_1, y_2, \dots, y_m$. Then name A_i the set of all functions $f : X \to Y$ such that $y_i \notin Im(f)$. These sets will be used for inclusion-exclusion and our result will be obtained as $m^n - |A_1 \cup A_2 \cup \cdots \cup A_m|$. How do we determine the number of functions in A_i for any i? Well, if a function misses y_i, that is basically a function from X to $Y \setminus \{y_i\}$, and there are $(m-1)^n$ such functions. Similarly one can count $A_i \cap A_j$ for $i \neq j$ as $(m-2)^n$ and so on and so forth. From here you just need to perform the actual inclusion-exclusion calculation and write down the sum. $\square$

Problem 10. *(HW) Use inclusion-exclusion principle to determine how many numbers below 100 are divisible by 2,3, or 5?*

Solution. We make three sets, A, B, C. Elements of A will be numbers below 100 divisible by 2. Elements of B will be numbers below 100 divisible by 3 and similarly for C and 5. Then it's easy to see that $A = \lfloor 100/2 \rfloor = 50$, $B = 33$, $C = 50$. To calculate the size of $A \cap B$ we need to find the number of numbers below 100 divisible by 6 which is $\lfloor 100/6 \rfloor$. To calculate the size of $A \cap C$ we need to find the number of numbers below 100 divisible by 10 and similarly for $B \cap C$. To find $|A \cap B \cap C|$ we need to find the number of numbers below 100 divisible by 30. Then the rest is a simple application of the inclusion-exclusion formula. $\square$

Problem 11. *(HW) Let X be a set of size n. How many distinct total orders can be defined over X?*

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[1] The version in the lecture was actually only a special case of the actual theorem. For general formulation see this link to the Wikipedia entry.

Solution. In total order every two elements are comparable. This means that we have a partial order of length n and width 0. In other words we have a chain. We can then choose n elements for the first element in the chain, $n-1$ for the second and soforth. Giving us a total on $n!$ total orders on a set of size n. $\square$

Problem 12. *(HW) Prove that every partial ordering on a finite sex X has at least one linear extension. For a partial ordering $\leq$, linear extension is any linear order $\preccurlyeq$ such that $x \preccurlyeq y \implies x \leq y$.*

Solution. We will proceed by induction. on the size of X. If X is a singleton then every partial order is a total order and we're done. For the inductive step assume that the statement holds for all sets of size n. Now assume that $|X| = n+1$. Let x_0 be the minimal element of X. Then consider the set $X' = X \setminus \{x\}$ with the parital order obtained by restricting the original partial order on X to X'. Then, as $|X'| = n$, we can assume that it admits a linear extension to this partial ordering. Now add x_0 back to X' in and adjust the linear order so that x_0 is the new minimum element. Then we have a linear ordering which is clearly an extension of the original partial ordering on X, as we wanted.

$\square$

Problem 13. *(HW) Determine that number of permutations of 1, 2, . . . , 8 in which no even integers are in their initial positions.*

Solution. Perform inclusion exclusion on sets A_2, A_4, A_6, A_8 where A_i is the set of all permutations of $\{1, 2, \ldots, 8\}$ fixing i. Size of each A_i is 7! since every permutation in A_i fixes i and permutes the other elements. And there are 4 $A_i's$. Size of each $A_i \cap A_j$ is 6! by similar logic and there are $\binom{4}{2}$ such pairs. Size of each $A_i \cap A_j \cap A_k$ is 5! and there is 4 of them. Size of $A_2 \cap A_4 \cap A_6 \cap A_8$ is 4!. The rest is pure computation using the inclusion-exclusion formula. $\square$