
Problem 1. *Let m, n, r be natural numbers such that $m \geq n \geq r$. Prove that $\binom{m}{r} \geq \binom{n}{r}$.*

Solution. These values correspond to choosing r elements from an m and n element sets respectively. In order to choose r elements from an m element set, we can ignore some $m-n$ elements of the set and choose the r elements from the remaining n elements, meaning that we have at least $\binom{n}{r}$ ways to do this, proving the result. □

Problem 2. *Find a way to place 8 rooks on a regular 8×8 chessboard in such a way that no two are attacking each other. Is it possible to place 9 rooks in such a way?*

Solution. To place 8 rooks one can just choose to put one rook on each square on the diagonal. Now assume that we have placed 9 rooks on the board in such a way that no two attack each other. It is easy to see that if two rooks are placed in the same column then they attack each other, therefore no two rooks can be placed in the same column. Now, since we have 9 rooks and 8 columns, two of the rooks need to be placed into the same column by the pigeonhole principle and therefore will attack each other. □

Problem 3. *Consider a regular deck of playing cards. How many ways are there to place 4 cards from the deck on the table? How many ways are there to place 4 cards on the table if the first card needs to be a red queen?*

Solution. In the first case there is $52 * 51 * 50 * 49$ ways to place the cards. We can place any card as the first one, hence 52 options, then we have 51 possible cards for the second card and so on. In the second case there is $2 * 51 * 50 * 49$ ways to place them, since there are only 2 cards that can be placed first. □

Problem 4.

1. *Let $X = \{1, 2, 3, 4\}$. How many ordered pairs (A, B) can you find, where $A, B \subseteq X$ and $A \cap B = \emptyset$.*
2. *Let $X = \{1, ..., n\}$ for some natural number n. How many ordered pairs (A, B) can you find, where $A, B \subseteq X$ and $A \cap B = \emptyset$.*

Solution. We will only prove the second part since the first part is just a concrete example. For each element we have three choices, it is either going to be placed in A, B or neither. So we have 3^n possible ways of doing this. □

Problem 5. *Give an algebraic and a combinatorial proof of the following:*

1. $\binom{n}{2} + \binom{n+1}{2} = n^2$
2. $\forall k, n \in \mathbb{N},\ k \leq n\ k\binom{n}{k} = n\binom{n-1}{k-1}$.

Solution. We will give the combinatorial proofs and leave the calculation to wolfram alpha or the student:

1. We claim that both sides count the number of pairs (j, k) such that $1 \leq j, k \leq n$. One can trivially see that the number of such pairs is n^2. On the other hand we can first count the choices where $j < k$, then $j = k$ and finally where $j > k$. To count the number of choices where $j < k$ is the same as two just choose two distinct numbers, there is $\binom{n}{2}$ ways to do this, similarly for $j > k$. Lastly, there is $\binom{n}{1}$ ways to choose j, k in such a way that $j = k$. Thus we have $\binom{n}{2} + \binom{n}{1} + \binom{n}{2} = \binom{n}{2} + \binom{n+1}{2}$ choices where the last identity is from Pascals triangle.

2. Assume that we have a class of n students and we want to choose a team of k students with a designated captain. One way to do this is to chose the captain and then choose the other $k-1$ members of the team. There is $n\binom{n-1}{k-1}$ ways to do this. The other possibility is to first choose the team and then pick the captain amongst the k team members. There is $k\binom{n}{k}$ ways to do this. Thus the equality is proven.

□

Problem 6. *Give combinatoiral proof of the following:*

1. $\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$ ***Hint: If we have a class of*** m ***girls and*** n ***boys, in how many ways can we choose a council of*** k ***people?***

2. $\sum_{m=k}^{n-k} \binom{m}{k}\binom{n-m}{k} = \binom{n}{2k}$. ***Hint: How many subsets of size*** $2k$ ***does the set*** $\{1, 2, \ldots, n\}$ ***have?***

Solution. Following the hint, one way to do is given by the identity on the right. Another way to count these subsets is to choose $2k$ people by choosing k from a subset of size $m \geq k$, and then k more from the rest. Summing up over all values of m gives the solution. □

Problem 7. ** Let p be a permutation of $\{1, 2, \ldots, n\}$ and let us write it in one line, for example (3 5 7 1 2 6 9 8). Now mark the increasing segments in this permutation, on our example we get (3 5 7—1 2 6—9—8). Let $f(n, k)$ denote the number of permutations of an n element set with exactly k increasing segments. Prove that $f(n, k) = f(n, n+1-k)$.*

Problem 8. *(HW) Consider the $n \times n$ grid with the bottom left point marked $(0, 0)$ and the top right point marked $(n-1, n-1)$. In how many ways can we reach $(n-1, n-1)$ from $(0, 0)$ by only moving right or up. That is, from a point (i, j) we are only allowed to move either to $(i+1, j)$ (moving right) or to $(i, j+1)$ (moving up).*

Solution. We need to make $2n-2$ steps, $n-1$ going UP and $n-1$ going RIGHT. Thus once we choose which of our steps are to the right, we are done. And there is $\binom{2n-2}{n-1}$ ways to choose the positions of these steps. □

Problem 9. *(HW) Let $F_0 = 0, F_1 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Let C_n be the number of ways to write n as a sum of 1's and 2's. For example $C_3 = 3$ since we can write $1+1+1 = 2+1 = 1+2 = 3$. Prove that $C_n = F_{n+1}$.*

Solution. Checking that $C_1 = F_2$ and $C_2 = F_3$ is simple and left to the reader. The important observation is that for every k, $C_k = C_{k-1} + C_{k-2}$. This holds because each sum of 1's and 2's summing to k starts with either a 2 or with a 1. If it starts with 1 then it is of the form $1 + (k-1)$. Similarly in the case if it starts with 2 it is of the form $2 + (k-2)$. Thus each sum of 1's and 2's adding up to k corresponds to either a sum adding up to $k-1$ or to a sum adding up to $k-2$. Vice versa if we have a sum of 1's and 2's adding up to $k-1$ or $k-2$ we can transform it into a sum adding up to k by adding a 1 or a 2 in front of it. Thus the relation holds. Now we have two sequences given by the same recurrence relation and they agree on the initial values. The result follows. $\square$