Discrete Math — Tutorial 2
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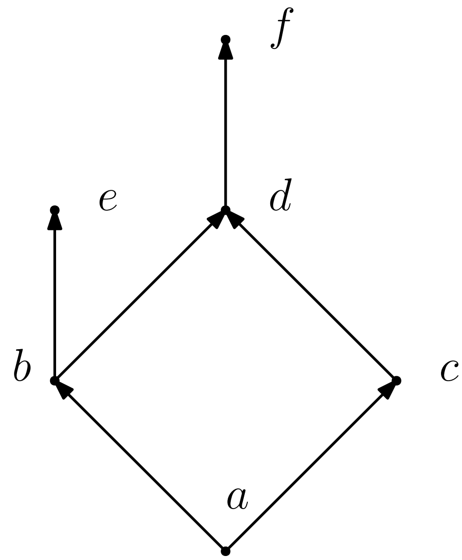
Problem 1. *The picture below can describes a relation R, where $(x, y) \in R$ means that there is a way to get from x to y using the arrows. Write down all the pairs which are in R.*



Solution. The following pairs are in R:

$$(a, c), (a, b), (a, e), (a, d), (a, f), (b, e), (b, d), (b, f), (c, d), (c, f), (d, f).$$

□

Problem 2. *Use induction (or strong induction) to prove the following:*

1. $\sum_{i=1}^{n} i(i+1) = \frac{1}{3}n(n+1)(n+2)$.
2. *If* $a_0 = 1$,$a_1 = 3$ *and* $\forall n \geq 2,\ a_n = 2a_{n-1} - a_{n-2}$ *then* $\forall n \geq 0, a_n = 2n + 1$

Solution. For the first part we will use weak and for the second part strong induction.

1. Base case $n = 1$: $\sum_{i=1}^{1} i(i+1) = 2 = \frac{1}{3}1(1+1)(1+2)$. Now we assume that statement holds for n and we want to prove that with this assumption the statement also holds for $n + 1$. So we have:

$$\sum_{i=1}^{n+1} i(i+1) = \sum_{i=1}^{n} i(i+1) + (n+1)(n+2) = \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2)$$

$$= (n+1)(n+2)(\frac{1}{3}n + 1) = \frac{1}{3}(n+1)(n+2)(n+3)$$

2. Base case $n = 2$: $a_2 = 2a_1 - a_0 = 5 = 2*2+1$ and $n = 3$: $a_3 = 2a_2 - a_1 = 7 = 2*3+1$. Now we assume that the statement holds for all integers until n for some $n \geqslant 3$. With this assumption we want to prove that statement holds for $n+1$. So we have $a_{n+1} = 2a_n - a_{n-1} = 4n + 2 - (2n - 2 + 1) = 2n + 3 = 2(n+1) + 1$.

□

Problem 3. *Consider the relations on people "is a brother of", "is a sibling of", "is a parent of", "is married to", "is a descendant of". Which of the properties of reflexivity, symmetry, antisymmetry and transitivity do each of these relations have?*

Solution. Reflexivity: none; Symmetry: "Is a sibling of", "Is married to"; Transitivity: "Is a sibling of", "Is a descendant of".

Note that if you define the word sibling as "two people are siblings if they have the same parent", then the relation becomes reflexive. □

Problem 4. *Let $f : X \to Y$, $g : Y \to Z$ be surjective functions. Prove that then $g \circ f : X \to Z$ is surjective as well. Find an example where $g \circ f$ is surjective but one of the functions f, g is not.*

Solution. Let $z \in Z$. We want to show that there is an $x \in X$ such that $g(f(x)) = z$. Since g is surjective we know that there exists a $y \in Y$ such that $g(y) = z$. Similarly there exists an $x \in X$ such that $f(x) = y$. Thus $g(f(x)) = z$. For the counterexample consider the following sets: $X = \{1, 2\}, Y = \{a, b, c\}, Z = \{\alpha\}$. Define $f : X \to Y$ by $f(1) = a, f(2) = b$, this is clearly not surjective. Define $g : Y \to Z$ as $g(y) = \alpha$ for every Y. Then the composition $g \circ f$ is celarly surjective while f isn't. □

Problem 5. *Give a relation R over the set of natural numbers $\mathbb{N}$ such that $R \setminus \{(i, i)|i \in \mathbb{N}\}$ is infinite and R is:*

1. *reflexive, symmetric and transitive.*
2. *reflexive, antisymmetric and transitive.*

Solution. We can use following two examples:

1. $R = \mathbb{N} \times \mathbb{N}$.
2. $R = \{(i, i)|i \in \mathbb{N}\} \cup \{(1, i)|i \in \mathbb{N}\}$.

□

Problem 6. *Let $f : X \to Y$ be a function and prove that:*

1. *f is injective if and only if there is a function $g : Y \to X$ such that $g \circ f$ is the identity on X.*
2. *f is surjective if and only if there is a function $g : Y \to X$ such that $f \circ g$ is the identity on Y.*

Solution. We only prove the first part as second is very similar. Suppose $f : X \to Y$ is injective, then by definition $f(x) = f(y)$ implies $x = y$ which means that if $y \in \operatorname{im}(f)$, then there exists a unique $x \in X$ such that $f(x) = y$. Define $g : Y \to X$ as follows: Fix $x \in X$ and define

$$g(y) = \begin{cases} f^{-1}(y) & \text{if } y \in \operatorname{im}(f), \\ x & \text{if } y \notin \operatorname{im}(f). \end{cases}$$

Observe that $g \circ f(x) = g\big(f(x)\big) = f^{-1}\big(f(x)\big) = x = \operatorname{id}_X(x)$ by injectivity of f and construction of g.

Conversely, suppose that there exists $g : Y \to X$ such that $g \circ f = \operatorname{id}_X$. Suppose $f(x) = f(y)$, then by hypothesis, we have $x = \operatorname{id}_X(x) = g \circ f(x) = g \circ f(y) = \operatorname{id}_X(y) = y$. Conclude by definition that f is injective. □

Problem 7. ** Let R be a relation on a set X. Prove that R is transitive if and only if $R \circ R \subseteq R$*

Problem 8. ** Let R, S be two equivalence relations on a set X. Prove that $R \circ S$ is an equivalence relation if and only if $R \circ S = S \circ R$*

Problem 9. *(HW) Let $f : A \to A$ be a function different from identity such that $f \circ f \circ f = f \circ f$. Prove that f is neither injective nor surjective.*
Hint: Use contradiction and Problem 6!

Solution. Assume that f is injective. By problem 6 there is a function $g : X \to X$ such that $g \circ f = id_X$. Then $f = g \circ g \circ f \circ f \circ f = g \circ g \circ f \circ f = id_X$ which contradicts our choice of f. Similar analysis works in the case where f is surjective but we need to use the existance of the right inverse from problem 6. □

Problem 10. *(HW) Let R, S be two equivalence relations on a set X. Which of the following are also equivalence relations? Prove your claims!*

1. $R \cup S$

2. $R \cap S$

Solution.

1. Let $X = \{a, b, c, d\}$,$R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a)\}$ and $S = \{(a, a), (b, b), (c, c), (d, d), (b, c$
   Then $(a, c), (c, b) \in R \cup S$ but $(a, b) \notin R \cup S$. So $R \cup S$ is not an equivalence relation.

2. We need to check that $R \cap S$ is reflexive,symmetric and transitive. For any $x \in X$, $(x, x) \in R$ and $(x, x) \in S$ since R, S are equivalence relations so $R \cap S$ is reflexive. Let $(a, b) \in R \cap S$. Then, by symmetry of R, $(b, a) \in R$ and similarly $(b, a) \in S$ so $R \cap S$ is symmetric. Finally if $(a, b), (b, c) \in R \cap S$. Then by transitivity of R and S, (a, c) is an element of both R and S and thus of $R \cap S$.

□

Problem 11. *(HW) Fibonacci numbers are defined as follows: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Prove the following:*

1. $\sum_{i=1}^{n} F_i = F_{n+2} - 1$

2. $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

Hint: Your proof should use induction!

Solution. We will prove only the first part as the second part is basically the same. Base case $n = 1$ is obviously true. Now assume that for every n, $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. Then $\sum_{i=1}^{n+1} F_i = \sum_{i=1}^{n} F_i + F_{n+1} = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$. And the proof is finished.
□