Discrete Math | Solutions to Tutorial 1
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Problem 1. *For each natural number t, let $X_t = (-t, t) = \{x \in \mathbb{R} : -t < x < t\}$ and $Y_t = [-t, t] = \{x \in \mathbb{R} : -t \leq x \leq t\}$. Describe the following sets:*

1. $\bigcup_{i=0}^{\infty} X_i$ *and* $\bigcup_{i=0}^{\infty} Y_i$
2. $\bigcap_{i=0}^{\infty} X_i$ *and* $\bigcap_{i=0}^{\infty} Y_i$

Solution.

1. We claim that $\bigcup_{i=0}^{\infty} X_i = \mathbb{R} = \bigcup_{i=0}^{\infty} Y_i$. It is obvious that $\bigcup_{i=0}^{\infty} X_i \subseteq \mathbb{R}$. For the other direction let $x \in \mathbb{R}$. Take the smallest natural number t such that $t > |x|$. Then $x \in X_t$ so $\mathbb{R} \subseteq \bigcup_{i=0}^{\infty} X_i$,finishing the proof. $\bigcup_{i=0}^{\infty} Y_i = \mathbb{R}$ can be proved similarly.
2. We claim that $\bigcap_{i=0}^{\infty} X_i = \emptyset$. This is because $X_0 = \emptyset$ and intersection of any set with the empty set is still empty. Similarly, since $Y_0 = \{0\}$ and $\forall t \geq 1$, $0 \in Y_t$ we can see that $\bigcap_{i=0}^{\infty} Y_i = \{0\}$.

□

Problem 2. *Let $A = \{2, 7, 15\}$ and B be a set such that $|B| = 5$ answer the following:*

1. *What is the smallest and largest cardinality (size) of $A \cap B$.*
2. *What is the smallest and largest cardinality (size) of $A \cup B$.*
3. *What is the smallest and largest cardinality (size) of $A \times B$.*

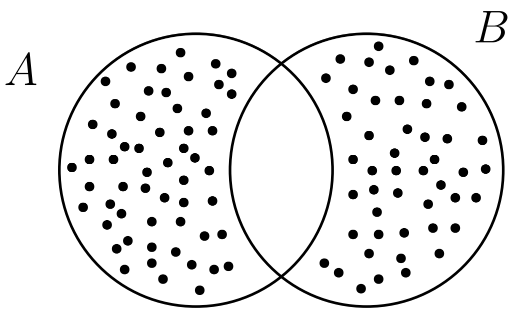
Solution.

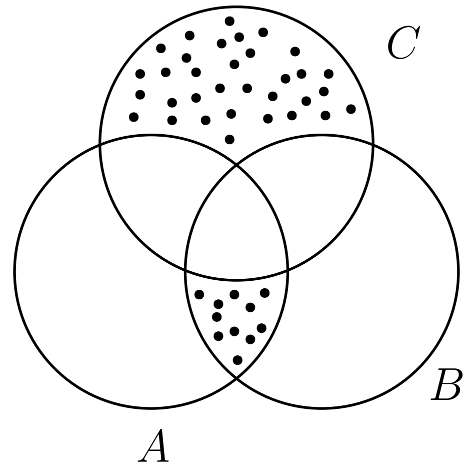
1. Smallest size of $A \cap B$ is 0, this happens when $B = \{a, b, c, d, e\}$ for example. Largest size is 3, occurring when $A \subseteq B$.
2. Smallest size of $A \cup B$ is 5, occurring whenever $A \subseteq B$. Biggest possible size is 8, occurring whenever $A \cap B = \emptyset$.
3. Smallest and largest size are the same in this case, since $|A \times B| = |A||B|$ for any two sets A and B.

□

Problem 3. *Find an example of a set A of cardinality 5 whose elements are sets and if $B \in A$ then $B \subseteq A$. (These sets are called transitive)*

Solution. One can construct transitive sets of arbitrary size using the following trick. In the process we also define the natural numbers using sets! Start by setting $0 := \emptyset$. Then let $1 := \{0\}$, $2 := \{0, 1\}$, $3 := \{0, 1, 2\}$, $4 := \{0, 1, 2, 3\}$ and $5 := \{0, 1, 2, 3, 4\}$. Obviously 5 has 5 elements and checking that each element is a subset is easy. □





Problem 4. *Describe the following Venn diagrams in set notation:*

Solution. For the left diagram the simplest description is $(A \cup B) \setminus (A \cap B)$. For the right diagram the following description works: $(C \setminus (A \cup B)) \cup ((A \cap B) \setminus C)$ □

Problem 5. *(**HW**) Let A, B, C be sets and prove the following:*

1. $(A, B \subseteq C) \implies (A \cup B \subseteq C)$
2. $(A = B) \iff (A \cup B = A \cap B)$
3. $(A \subseteq B) \implies ((A \cup C) \subseteq (B \cup C))$

Solution. Note that some authors use "$A \subset B$" to mean "A is a subset of B" while some other authors use it to mean "A is a porper subset of B", that is, A is a subset of B but not equal to B. The statements above are incorrect if "$\subset$" symbol is used with the latter interpretation (proper subset). For example, $A = \{1\}, B = \{2\}, C = \{1, 2\}$ contradicts the first claim. Here to avoid any ambiguity we will use "$\subseteq$".

1. Suppose $A, B \subseteq C$. Now, let $x \in A \cup B$. Then either $x \in A$ or $x \in B$. Either way, since $A \subseteq C$ and $B \subseteq C$, we can conclude that $x \in C$ so $A \cup B \subseteq C$.

2. For the " $\implies$ " direction, assume $A = B$. Then $A \cap B = A \cap A = A = A \cup A = A \cup B$. For the " $\impliedby$ " direction, assume $A \neq B$ for contraction. Without loss of generality[1] we can then assume that there is an $x \in A \setminus B$. Then clearly $x \in A \cup B$ but $x \notin A \cap B$. Hence $A \cup B \neq A \cap B$, so we reached a contradiction.

3. Let $x \in A \cup C$. If $x \in A \setminus C$ then $x \in B$ since $A \subseteq B$ and hence we conclude that $x \in B \cup C$. If $x \in C$ then obviously $x \in B \cup C$ by definition.

□

Problem 6. *Let $A, B \subset \mathbb{N}$ be sets of natural numbers and prove the following:*

[1] If there is no such x then there must be $x \in B \setminus A$ otherwise $A = B$ contrary to our assumption. We can switch the names of the sets and proceed with the (counter)example where $x \in A \setminus B$.

1. $(A \cup B)^c = A^c \cap B^c$

2. $(A \cap B)^c = A^c \cup B^c$

Solution.

1. $x \in (A \cup B)^c \iff x \notin A$ and $x \notin B \iff x \in A^c$ and $x \in B^c \iff x \in A^c \cap B^c$.

2. $x \in (A \cap B)^c \iff x \notin (A \cap B) \iff x \notin A$ or $x \notin B \iff x \in A^c \cup B^c$

□

Problem 7. *Let A be a set and for some $n \in \mathbb{N}$ consider the collection of subsets of A denoted by $X_1, \ldots, X_n$. Then prove the following:*

1. $A \setminus \bigcup_{i=1}^n X_i = \bigcap_{i=1}^n (A \setminus X_i)$

2. $A \setminus \bigcap_{i=1}^n X_i = \bigcup_{i=1}^n (A \setminus X_i)$

How does this exercise relate to the previous one?

Solution. The argument for proving this is essentially the same as the previous one. Simply replace n with 2 and A with $\mathbb{N}$. Then repeat the same proof as above :). □

Problem 8. ***(HW)*** *Prove or Disprove: Let X, Y, Z be sets, if $X \times Y = Z \times Y$ then $X = Z$.*

Solution. Based on interpretation of the problem there are two acceptable solutions:

1. Without additional assumption statement is false if $Y = \emptyset$. In this case $X \times Y = Z \times Y = \emptyset$ for any choice of X, Z so they can be different.

2. Assume $Y \neq \emptyset$. In this case one can argue by contradiction. Assume $X \neq Z$. Then, without loss of generality we can assume that there exists $x \in X \setminus Z$. Then $\forall y \in Y$, $(x, y) \in X \times Y \setminus Z \times Y$ so we reach a contradiction (Notice that if $Y = \emptyset$ then this last argument is not justified because there are no elements $y \in Y$ that would provide an ordered pair (x, y) to be used for contradiction).

□