
Problem 1. *Use induction to prove the following statements:*

1. *(HW)* $\forall n \in \mathbb{N}, 5^n - 1$ *is divisible by* 4.
2. $\forall n \in \mathbb{N}, 2^n \leq (n+1)!$.

Problem 2. *Let a relation* $\sqsubseteq$ *on* $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ *be defined as* $(a, b) \sqsubseteq (c, d)$ *if* $ad \leq cb$. *Is* $\sqsubseteq$ *a partial order?*

Problem 3. *Let* $a_1, ..., a_n$ *be* n *integers which are not necessarily distinct. Prove that there is always a set of consecutive numbers* $a_k, a_{k+1}, ..., a_l$ *whose sum is a multiple of* n. *Hint: Define a function sending* $m \in \{1, ..., n\}$ *to the remainder of* $\sum_{i=1}^{m} a_i$ *when divided by* n. *Then use pigeonhole principle.*

Problem 4. *Prove that* $E[X^2] \geq (E[X])^2$ *holds for any random variable* X.

Problem 5. *Euler's formula* $(v - e + f = 2)$ *holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of* $v - e + f$ *now? What if it has* k *components?*

Problem 6. *Let* R *be a relation over a set* X. *The symmetric closure of* R *is the relation* $R \cup R^{-1}$. *The transitive closure of* R *is the smallest superset of* R *that is transitive.*

1. *Prove that the symmetric closure of* R *is the smallest superset of* R *that is symmetric.*
2. *Prove that the transitive closure of* R *is* $\bigcup_{i=1}^{\infty} R^i$, *where* $R^1 = R$ *and* $R^{i+1} = R \circ R^i$.
3. *Prove that the transitive closure of a symmetric relation is symmetric.*
4. *Prove that the symmetric closure of a transitive relation need not be transitive.*

Problem 7. *Prove that if you color the edges of* K_6 *in red and blue, you are guarteed to have a monochromatic triangle.*

Problem 8. *Let* $\mathcal{R}_n$ *be the set of all relations over the set* $[n]$. *A relation* R *is picked from* $\mathcal{R}_n$ *uniformly at random.*

1. *What is the probability that* R *is reflexive?*
2. *What is the probability that* R *is symmetric?*
3. *(HW) Are the events "*R *is reflexive" and "*R *is symmetric" independent?*
4. *(HW) Define the set* $E_R := \{\{i\} \cup \{j\} \mid (i, j) \in R\}$. *What is the probability that* $([n], E_R)$ *is a (simple undirected) graph?*
5. *(HW) Define* $X : \mathcal{R}_n \to \mathbb{N}$ *as follows:*

$$X(R) = \begin{cases} |R|, & \textit{if } R \textit{ is reflexive} \\ 0, & \textit{otherwise} \end{cases}$$

   What is the expected value of X*?*