Discrete Math — Tutorial 9

Todor Antić/Hans Raj Tiwary — HW Due: 20.12.2023

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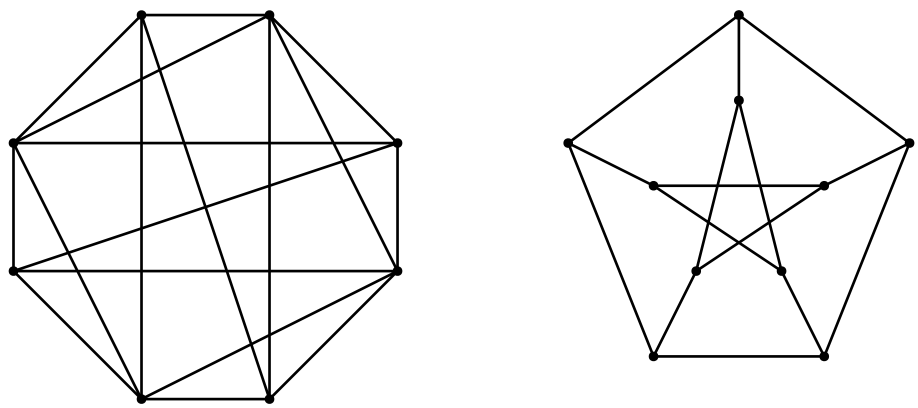
**Problem 1.** *Three unfriendly neighbours use the same water, beer and food sources. In order to avoid meeting, they wish to build non-crossing paths from each of their houses to each of the three sources. Can this be done?*

**Problem 2.**

1. *Draw two non-isomorphic planar graphs with the same number of vertices, edges, and faces.*
2. *Draw two planar graphs with the same number of vertices and edges, but different number of faces.*
3. *Can the graphs in the previous question be isomorphic?*

**Problem 3.** *Draw the below graphs with as few crossings as possible.*



**Problem 4.** *For a graph $G$, we define the line graph of $G$, $L(G)$ to be the graph such that $V(L(G)) = E(G)$ and $e, e'$ form an edge in $L(G)$ if $e \cap e' \neq \emptyset$. Prove that if $G$ is connected then so is $L(G)$.*

**Problem 5.** *For any natural number $n$ define the graphs $H_n = (V_n, E_n)$ as follows:*

$$V_n = \{0, 1, \ldots, 2^n - 1\},$$

$$E_0 = \emptyset, E_{n+1} = E_n \cup \{\{2^n + i, 2^n + j\} \mid \{i, j\} \in E_n\} \cup \{\{i, 2^n + i\} \mid 0 \leqslant i \leqslant 2^n - 1\}.$$

1. *Draw $H_n$ for $n = 3$.*
2. *For which values of $n$ is $H_n$ planar.*

**Problem 6.** *Let $\mathcal{G}$ be a set of graphs such that for no two distinct $G, H \in \mathcal{G}$ are isomorphic to each other. Let $\preceq$ be a relation over $\mathcal{G}$ defined as follows: $H \preceq G$ iff $H$ is a minor of $G$. Prove that $(\mathcal{G}, \preceq)$ is a poset.*

**Problem 7.** *A graph $G$ is called* outerplanar *if it can be drawn in the plane in such a way every vertex of $G$ lies on the outer face.*

1. *Prove that $K_4$ and $K_{2,3}$ are planar but not outerplanar.*

2. *Prove that every outerplanar graph contains a vertex of degree 2 or less.*

**Problem 8.** *Prove that in each drawing of $K_n$ for $n > 5$, there is at least $\frac{1}{5}\binom{n}{4}$ crossings. Use the non-planarity of $K_5$*

**Problem 9.** *(HW) Prove that there is a number $n_0$ such that for any graph with $n \geqslant n_0$ vertices, either $G$ or $\overline{G}$ is not planar.*

**Problem 10.** *(HW) Characterize all values of $m, n$ such that $K_{m,n}$ is planar.*