
Problem 1. *Let (Ω, P) be a discrete probability space and let $X : \Omega \to \mathbb{R}$ be a random variable. Prove that if $\mathbb{E}[X] < a$ for some $a \in \mathbb{R}$, then there exists an outcome $\omega \in \Omega$ such that $X(\omega) < a$.*

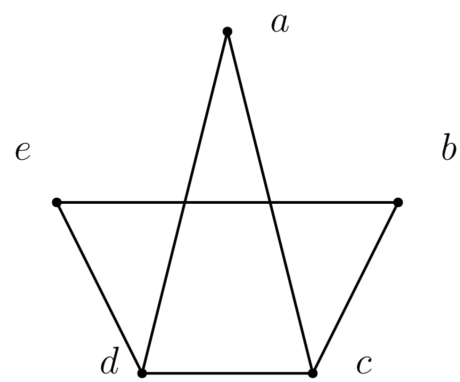
Problem 2. *Let $G = (V, E)$ be a graph. The* degree *of a vertex $v \in V$ – denoted by $\delta_G(v)$ – is the number of edges incident to v. In other words,*

$$\delta_G(v) := |\{e \in E \mid v \in e\}|.$$

The average degree of a graph G is defined to be the average of the degrees of the vertices, that is, $\dfrac{\sum_{v \in V(G)} \delta_G(v)}{|V(G)|}$. Compute the average degrees of $K_n, K_{m,n}, P_n$, and C_n.

Problem 3. *Among a group of 5 people, is it possible for everyone to be friends with exactly 2 of the people in the group? What about 3 of the people in the group?*

Problem 4. *Is the graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$, and $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$ equal to the graph H drawn below? Are they isomorphic?*



Problem 5. *Suppose a subset of $[n]$ is picked uniformly at random. Consider the random variable $X : 2^{[n]} \to \mathbb{R}$ with $X(S) = \max S$. For example, $\max\{3, 2, 6, 4\} = 6$. Compute the expected value of X*

Problem 6. *Show that there is a way to color the edges of K_n, $n > 5$ with two colors in such a way that there is at most $\binom{n}{5} 2^{1-\binom{5}{2}}$ monochromatic copies of K_5. (Hint: Use linearity of expectation!)*

Problem 7. *(*) Let G be a graph on n vertices. Let d be it's average degree, Δ the maximal degree and α the size of the largest independent set (induced subgraph with no edges). Then prove the following:*

1. $\alpha(G) \geq \frac{n}{\Delta+1}$
2. $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{deg(v)+1}$

Problem 8. *(HW) The complement of a graph* $G = (V, E)$ *is the graph* $\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$. *Show that two graphs are isomorphic if and only if their complements are isomorphic.*

Problem 9. *(HW) We call a graph* G *self-complementary if it is isomorphic to its complement* $\overline{G}$. *Find all self-complementary cycles and prove that no others exist.*