Discrete Math | Tutorial 6
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Problem 1. *List all graphs with vertex set* $\{1, 2, 3, 4\}$ *and two edges.*

Problem 2. *Let* X_n *be the sum of numbers obtained by rolling the dice* n *times, compute* $\mathbb{E}(X_n)$.

Problem 3. *(Bayes' Theorem) Let* (Ω, P) *be a finite probability space,* A *an event, and* $B_1, \ldots, B_n$ *a partition of* Ω.

1. *Prove that* $P[A] = \sum_{i=1}^{n} P[A \cap B_i]$.

2. *Let* $i \in \{1, \ldots, n\}$. *Prove that* $P[B_i|A] = \frac{P[A|B_i] \cdot P[B_i]}{P[A]}$.

3. *Let* $i \in \{1, \ldots, n\}$. *Prove that*

$$P[B_i|A] = \frac{P[A|B_i] \cdot P[B_i]}{\sum_{j=1}^{n} P[A|B_j] \cdot P[B_j]}.$$

Problem 4. *A box contains three coins: two regular coins and one fake two-headed coin* $(P(H) = 1)$,

1. *You pick a coin at random and toss it. What is the probability that it lands heads up?*

2. *You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?*

Problem 5. *Is it possible to have a tournament where for every subset* S *of* k *players, there exists another player who has defeated every player in* S*?*

Let T *be a random tournament with* n *players* $P = \{p_1, \ldots, p_k\}$ *where for every pair* p, q *of players the winner is decided by the toss of a fair coin.*

1. *Let* $S = \{p_1, \ldots, p_k\}$, *and let* $w \notin S$. *What is the probability that player* w *is defeated by some player in* S*?*

2. *Let* $S = \{p_1, \ldots, p_k\}$, *and let* $w \notin S$. *What is the probability that for every player* $w \notin S$, w *is defeated by some player in* S*?*

3. ** Prove that the probability that a random tournament does not meet the requirement stated at the beginning of the problem is strictly less than 1 for large enough* n.

Problem 6. *Seat* b *black and* r *red knights at random around a round table on chairs numbered counterclockwise from 1 to* $n = b + r$ *so that all seating arrangement are equaly likely. Let* X *be the number of black knights that have a black knight to their right. Compute* $\mathbb{E}(X)$.

Problem 7. ** In an $n \times n$ array, each of the numbers $1, 2, \ldots, n$ appears exactly n times. Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct numbers.*

Problem 8. ** A fair coin is tossed until two consecutive tails appear. The tosses are independent. Denote by N the number of necessary tosses including the last two tails. Express the probabilities $P(n = N)$ for $n \geq 2$ in terms of Fibonacci sequence $\{F_n\}_{n\geq 1}$ given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$.* ***Hint:*** *condition on first two tosses and derive a recursive formula for the sequence $a_n = 2^n P(N = n)$.*

Problem 9. *Prove that there is a constant c such that whenever we have two natural numbers n, m such that $n > cm^2$ then a random mapping $f : \{1, 2, ..., n\} \to \{1, 2, ..., m\}$ is surjective with probability at least 0.99. IMPORTANT: You will need to use the fact that for any $p \in (0, 1)$, $1 - p \leq e^{-p}$.*

Problem 10. *(HW) Let (Ω, P) be a finite probability space, and let $A_1, A_2, \ldots, A_n$ be events. Prove that*

$$P[\bigcup_{i=1}^{n} A_i] = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} P[\bigcap_{i\in I} A_i].$$

Problem 11. *(HW) Prove that there is a constant c such that whenever we have two natural numbers n, m such that $m > cn^2$ then a random mapping $f : \{1, 2, ..., n\} \to \{1, 2, ..., m\}$ is injective with probability at least 0.99.*