Discrete Math
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Problem 1. Give an example of a partial order on X where X has 15 elements such that width of your partial order is 3 and length is 5.

Problem 2. Your favorite mathematical pizza chain offers 10 toppings. How many pizzas can you make if you are allowed 6 toppings? The order of toppings does not matter but now you are allowed repeats. So one possible pizza is triple sausage, double pineapple, and onions.

Problem 3. Describe all relations on a set X that define an equivalence relation and a partial ordering at the same time.

Problem 4. Let R be a relation over a set X, and let $Y \subseteq X$.

- 1. Prove that if R is an ordering over X then $R \cap (Y \times Y)$ is an ordering over Y.
- 2. Prove that if R is an equivalence relation over X then $R \cap (Y \times Y)$ is an equivalence relation over Y.

Problem 5. Consider a poset (P, \leq) where $P = \{1, 2, 3, 4, 5, 6\}$ and the partial order is defined by the divisibility relation ($a \leq b$ if and only if a divides b). Draw the Hasse diagram for this poset.

- 1. Identify all maximal and minimal elements of the poset.
- 2. Find the length of the longest chain and the width of the poset.
- 3. Verify that for this poset, $\alpha(P)\omega(P) \ge |P|$, where $\alpha(P)$ is the size of the largest antichain and $\omega(P)$ is the size of the largest chain.

Problem 6. Prove that every partial ordering on a finite set X has at least one linear extension. For a partial ordering \leq , linear extension is any linear order \preccurlyeq such that $x \preccurlyeq y \implies x \leq y$.

Problem 7. Use double counting to prove the following:

1.
$$\forall n \ge s \ge r \ge 0$$
, $\binom{n}{s}\binom{s}{r} = \binom{n}{r}\binom{n-r}{s-r}$

2.
$$\forall n \ge 1, \sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$$

Problem 8. (*HW*) In a group of 120 students, each student is assigned a unique ID number from 1 to 120. We define an equivalence relation on this set of students as follows: two students are related if and only if the difference between their ID numbers is divisible by 4. How many equivalence classes does this relation create, and what is the size of each equivalence class?

Problem 9. (*HW*) Consider the set $X = \{f : \mathbb{N} \to \mathbb{R}\}$ and a relation \leq on X defined by $f \leq g$ if there exists an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $f(n) \leq g(n)$. Show that this relation is a partial order but not a linear order.

Problem 10. (*HW*) We say that two posets (A, \leq) and (B, \preceq) are isomorphic if there is a bijection $f: A \to B$ such that $x \leq y \implies f(x) \preceq f(y)$ for all $x, y \in A$. Prove that:

- 1. Every two linear orderings on an n-element set are isomorphic.
- 2. There exist two linear orderings on \mathbb{N} which are not isomorphic.