Exercise 3: Discrete & Continuous Optimization

1. Approximate $\sqrt{2}$ using Newton's method for root-finding. Use $x_0 = 1$ as the initial approximation and iterate the method four times.

2. Find the minimum point of $f(x) = e^{-x} + x^2$ using Newton's method. Use $x_0 = 1$ as the intial value and stop when the solution changes by less than 0.01.

3. Consider the following unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^2} f(x) = x_1^4 + x_2^4 - 2x_1 x_2$$

   (a) Find the gradient and the Hessian of f.

   (b) Use the Newton's method to find an approximate solution starting from $x_0 = (1, 1)$.

   (c) Explain why it is more efficient to solve the system of linear equations $\nabla^2 f(x_k) y = \nabla f(x_k)$ than to invert the matrix $\nabla^2 f(x_k)$ at each iteration.