Exercise 2: Discrete & Continuous Optimization

1. Show that the following functions are convex or not convex on their domains using the test of Hessian being positive semidefinite:

   (a) $f(x,y) = x^3 + y^3 - 3xy$ on $\mathbb{R}^2$

   (b) $g(x,y) = x^2 + 2xy + y^2$ on $\mathbb{R}^2$

   (c) $h(x,y,z) = x^2 + y^2 + z^2 - 2xyz$ on $\mathbb{R}^3$

2. Let M be a convex set. Recall that a function $f : M \to \mathbb{R}$ is said to be convex over M if for all $x_1, x_2 \in M$ and for all $\lambda \in [0,1]$ we have that $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$. Prove that $f : M \to \mathbb{R}$ is convex over M if and only if for any $x_1, \ldots, x_k \in M$ and $\lambda_1, \ldots, \lambda_k \geqslant 0, \sum_{i=1}^{k} \lambda_i = 1$ we have that

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \leqslant \sum_{i=1}^{k} \lambda_i f(x_i).$$

3. Let P be a symmetric $n \times n$ matrix and let $c, d \in \mathbb{R}^n$. Prove that $f(x) = x^\intercal P x + c^\intercal x + d$ is convex if and only if P is positive semidefinite.

4. Prove that $f(x) = -\sum_{i=1}^{m} \ln(b_i - a^\intercal x)$, is convex over $\{x | a_i^\intercal x < b_i, i = 1, \ldots, m\}$.