KT graph orientations

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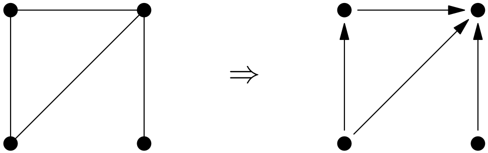
Orientation of a digraph

Definition

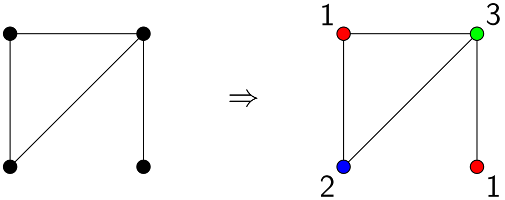
Let $G = (V, E)$ be a graph. We say $H = (V', E')$ is an *orientation* of G if $V' = V$ and for all $(x, y) \in E$ either $(x, y) \in E'$ or $(y, x) \in E'$. A *digraph* is a graph with an orientation.



Coloring of a graph

Definition

A *(proper) vertex n-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, ..., n\}$ such that for all $(x, y) \in E$, $f(x) \neq f(y)$.



Parameters of a graph

Definition (Chromatic number)

The *chromatic number* of a graph G (denoted $\chi(G)$) is the minimum number of colors required to obtain a proper coloring of G.

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Definition (Clique number)

The *clique number* of a graph G (denoted $\omega(G)$) is the number of vertices in a maximum clique (subgraph in which every pair of vertices have an edge) of G.

Background

Observation

$\chi(G) \geq \omega(G)$.

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Definition (χ-boundedness)

A graph G is χ-*bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(H) \leq f(\omega(H))$ for each induced subgraph H of G.

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Examples

Perfect graphs, i.e., graphs G for which $\chi(G) = \omega(G)$. (Eg: Triangle graph K_3.)

History

Question

Are all graphs χ-bounded?

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NO! Erdős, Mycielski, Tutte (separately) constructed graphs G with large girth and large chromatic number, i.e., $\omega(G) \leq 2$, and $\chi(G) = t$, for any $t \in \mathbb{N}$.

History

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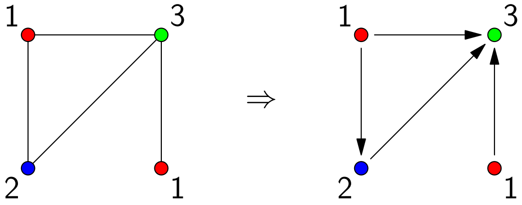
Conjecture (Gyárfás-Sumner)

All forests are χ-bounded.

For the digraph variant of the Gyárfás-Sumner conjecture, Kierstead and Trotter considered the following orientation:

Definition

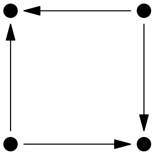
Let G be a graph. The *natural orientation* of G is the colored digraph $NG = (V, A, f)$, with arc set $A = \{(x, y) : xy \in E$ and $f(x) < f(y)\}$.



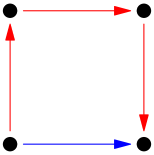
KT orientation

Definition

Let G be a graph, and D be an orientation of G. We say that D is a *KT-orientation* if for all u, v in $V(G)$, D contains at most one directed path between u and v.



YES



NO

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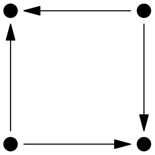

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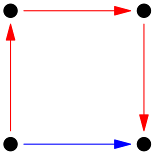


NO

Observation

D contains no directed cycle!

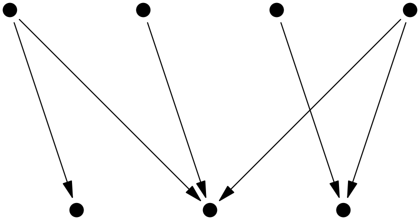
Problem

Problem

Which graphs G have a KT-orientation?

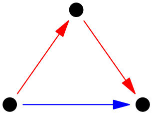
Basic examples

Bipartite graphs.



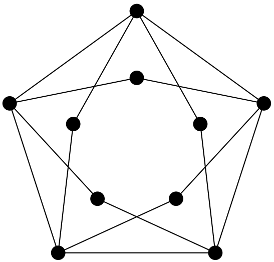
Basic non-examples

Graphs containing K_3.



Basic non-examples

Grötzsch graph without a vertex.



To find more non-examples and the underlying pattern for classifying the graph families admitting a KT orientation.

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The KT orientations have already found applications in:

- Constructing a counterexample to a conjecture about triangle-free induced subgraphs of graphs with large chromatic number.
- Separating polynomial χ-boundedness from χ-boundedness.

References

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Acknowledgement

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Thank you for listening!