3rd problem set for Probability and Statistics — March 11/12

Independent events

Reminder: If I is an arbitrary set of indices, the events A_i for $i \in I$ are *independent* if for every finite set $J \subseteq I$

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j).$$

If the condition holds only for two-element sets J, we call the events $\{A_i : i \in I\}$ pairwise independent.

1. (a) We will model two coin flips by the uniform probability space $\{HH, HT, TH, TT\}$. Verify that the events "first flip came up heads" and "second flip came up heads", denoted A_1 and A_2 respectively, are independent.

(b) We again have a probability space with four elementary events HH, HT, TH, and TT, but this time it is not uniform. As in the previous example, A_1 is the event "first letter is H" and A_2 is the event "second letter is H". We assume that $P(A_1) = p_1$, $P(A_2) = p_2$, and that events A_1 and A_2 are independent. Verify that this determines the probability of each event in this probability space.

2. Prove that if events A, B are independent, then so are the events A, B^c. And the same holds for events A^c, B^c.

3. (a) Is it possible for events A, B to be independent and disjoint at the same time?
(b) Is it possible for events A, B to be independent and at the same time $A \subseteq B$?

4. Find some events A, B, C (in any probability space) that
(a) are independent.
(b) are not pairwise independent but $P(A \cap B \cap C) = P(A)P(B)P(C)$.
(c) are pairwise independent but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

5. * Can you find some events A, B, C (in any probability space) such that

- A, B are independent given C, i.e.,

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C),$$

- A, B are independent given C^c,
- but A, B are not independent.

Random variables

6. Three friends decide to go swimming on some day of a given week but don't fix the day. So, each one shows up at the swimming pool on a (uniformly) random day, independently. Consider the random variable X to be the number of friends (from these three) who went on Friday. Find the probability distribution of X. Generalize this to n friends.

7. Adam shoots a basketball at a basket, on each trial he has probability of hitting 1/10, the trials are independent. He quits after the first hit. Let X denote the total number of shots.

(a) What is $P(X > k)$? (Try first for $k = 1$, $k = 2$.)

(b) What is the distribution of X? That is, determine the probability function p_X, i.e. for each x determine $P(X = x)$ (do you know the name of this distribution?).

(c) What is $P(X \geq 10 \mid X \geq 5)$?

8. Continuing from the last problem: let's denote $Y = X \bmod 2$, i.e. $Y = 0$ if X is even, otherwise $Y = 1$. Determine the distribution of Y.

9. Beatrice also shoots a basketball, she has probability p of hitting the basket. Let Z denote the number of hits from n attempts. Determine the distribution of Z.

10. Let X and Y be discrete random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that $f(X)$ and $X + Y$ are discrete random variables.

Bonus problems

11. (St. Petersburg Casino) We flip a coin repeatedly. If the first time the coin comes up heads on the nth flip, we get a reward of 2^n rubles. How much would you be willing to pay to participate in this game?

More practice problems

12. There are a hundred coins in the chest. 99 of them are normal, but one has eagle on both sides. We pull out a random coin and flip it six times, each time it lands tails. What is the probability that that we've pulled a two-eagle coin? (Try to guess first, then calculate.)

13. For disease C we have two tests, A and B. The A test has both sensitivity and specificity $p = 0.95$. The B test always says the patient is healthy. Assume that $P(C) = 0.01$.

(a) Calculate for both tests the probability of success (i.e., the correct answer) if we use them to a random patient. Think about what this says about the usefulness of both tests.

(b) For what p is the probability of success of both tests the same?

14. In an election, people vote for two candidates, A and B. When leaving the polling station voters are randomly asked to participate in an exit-poll. Assume that whoever answers will answer truthfully who they voted for, but not everyone will participate. If we denote by E the set of voters who participate in the exit-poll, then suppose $P(E \mid A) = 0.7$ and $P(E \mid A^c) = 0.4$. The exit-poll results are 60 % for A. What is the actual proportion of people who voted for A?

15. We use smoke signals to transmit a binary file. Therefore, there is a relatively high probability of error for each bit: 0 is transmitted as 0 only with probability 0.9, 1 is transmitted as 1 only with probability 0.8. Assume (somewhat unreasonably) that each character is transmitted independently. Assume further that there are equal numbers of zeros and ones in the transmitted message.

(a) If we received a 0 signal, what is the probability that it was actually sent?

(b) We received message 0010. What is the probability that it was actually sent?

(c) How does the calculation change if we send each symbol three times to check (and then take the more frequent of the three attempts)?