10th problem set for Probability and Statistics — May 6/7

Summary

- **Method of Moments Estimation** solves the equation $m_1(\theta) = \widehat{m}_1(\theta)$ for the unknown θ.
- Or a system of equations $m_r(\theta) = \widehat{m}_r(\theta)$ for $r = 1, 2, \ldots$ as needed.
- $L(\theta; x_1, \ldots, x_n) = P(X_1 = x_1 \,\&\ldots\&\, X_n = x_n)$ ... likelihood of observed data dependent of parameter θ.
- or $L(\ldots) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ ... probability density function ...
- $\ell(\theta; x_1, \ldots, x_n) = \log L(\ldots)$ ... for easier computations.
- **Maximum Likelihood Estimation (MLE)** searches for θ for which $L(\theta; x_1, \ldots, x_n)$, or $\ell(\ldots)$, is maximized. Usually done using derivatives of L, or ℓ.
- **Bias:** $\mathbb{E}_\theta(\hat{\theta} - \theta)$ ... θ true parameter, $\hat{\theta}$ our estimate (random variable as it depends on observed data).
- **MSE (Mean Square Error):** $\mathbb{E}_\theta((\hat{\theta} - \theta)^2)$
- Theorem: $MSE = bias(\hat{\theta})^2 + \mathrm{var}(\hat{\theta})$.

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1. We have a random sample $X_1, \ldots, X_n \sim U(0, \vartheta)$.
(a) Propose a point estimate ϑ using the method of moments.
(b) Propose a point estimate ϑ using the method of maximum likelihood.
(c) For each of them, determine whether it is unbiased and consistent.
(d) Compute the Mean Square Error (MSE) for each of them.
(e) Which estimate is better? Can you think of an even better one?

2. For a random sample $X_1, \ldots, X_n \sim Geom(p)$.
(a) Propose a point estimate ϑ using the method of moments.
(b) Propose a point estimate ϑ using the method of maximum likelihood.
(c) For each of them, determine whether it is unbiased and consistent.

3. For a random sample $X_1, \ldots, X_n \sim Exp(1/\theta)$.
(a) Propose a point estimate θ using the method of moments.
(b) Propose a point estimate θ using the method of maximum likelihood.
(c) For each of them, determine whether it is unbiased and consistent.
(d) Compute the Mean Square Error (MSE).

4. We have a random sample $X_1, \ldots, X_n \sim Exp(\lambda)$. We are interested in the probability p that $X > 1$ for $X \sim Exp(\lambda)$. (Recall that $p = e^{-\lambda \cdot 1}$.)
(a) Propose a point estimate for p (using any method), or several estimates if desired.
(b) Investigate its properties.

5. Let $S = \sum_{k=0}^{30} \binom{100}{k}$. Also, let $X = \sum_{i=1}^{100} X_i$, where X_i is 0 or 1, both with probability 1/2 and the variables $X_1, \ldots, X_n$ are independent. Thus, $X \sim Bin(100, 1/2)$.
(a) Express S using the cumulative distribution function F_X.
(b) Use CLT to estimate this probability.
(c) Alternatively, compute S using appropriate software and compare.

Additional Exercises

6. Let $X \sim Exp(\lambda)$ describe the distance traveled by a radioactive particle before decay. Our device captures its decay (and the decay location, i.e., the value of X), but only if $1 \leq X \leq 2$. Formally, we will consider a random sample $X_1, \ldots, X_n \sim F_{X|B}$ for the event $B = 1 \leq X \leq 2$.

(a) Propose a point estimate λ using the method of moments.
(b) Propose a point estimate λ using the method of maximum likelihood.
(c) For each of them, determine whether it is unbiased and consistent.

7. We have a random sample $X_1, \ldots, X_n \sim Pois(\lambda)$.

(a) Propose a point estimate for λ using the method of moments.
(b) Propose a point estimate for λ using the method of maximum likelihood.
(c) Calculate the mean squared error (MSE).

Interval estimates (NOT for the tutorial test)

8. We have one measurement $X \sim N(\mu, 1)$. (That is, the parameter $\vartheta = \mu$.)

(a) Find a 95% confidence interval for μ. (For specificity: we measured $x = 2.9$.)

(b) Instead of one measurement, we perform n (of course, independent) measurements. What will be the confidence interval for μ now? For specificity: we measured $x_1, \ldots, x_9 =$ 1.82, 1.00, 2.50, 3.00, 0.50, 2.97, 1.76, 1.35, 3.41.

(c) Let X still have mean μ and variance 1, but it is no longer necessarily normal. What changes?

9. This time we are sampling from $N(\mu, \sigma^2)$: we do not know μ or σ, so the parameter is $\vartheta = (\mu, \sigma)$. We measured values 8.47, 10.91, 10.87, 9.46, 10.40.

(a) Calculate the sample mean and sample variance.

(b) If we believed that the computed sample variance is the true value of σ^2, find a confidence interval for μ.

(c) Find a confidence interval for μ using the Student's t-distribution.

10. We model the number of emails per day using a Poisson distribution $Pois(\lambda)$. In the first week of May, we received 34, 35, 29, 31, 30 emails. Find a 95% confidence interval for λ.

Use the last method from the lecture – the one using the Student's distribution. (Although the Poisson distribution is not normal, for sufficiently high values of λ, it is very similar to normal, so the method will have reliability close to 95%.)

p	0.9	0.95	0.975	0.99	0.995
$\Phi^{-1}(p)$	1.28	1.64	1.96	2.33	2.58
$\Psi_1^{-1}(p)$	3.08	6.31	12.71	31.82	63.66
$\Psi_2^{-1}(p)$	1.89	2.92	4.3	6.96	9.92
$\Psi_4^{-1}(p)$	1.53	2.13	2.78	3.75	4.6
$\Psi_8^{-1}(p)$	1.4	1.86	2.31	2.9	3.36
$\Psi_{20}^{-1}(p)$	1.33	1.72	2.09	2.53	2.85
$\Psi_{30}^{-1}(p)$	1.31	1.7	2.04	2.46	2.75