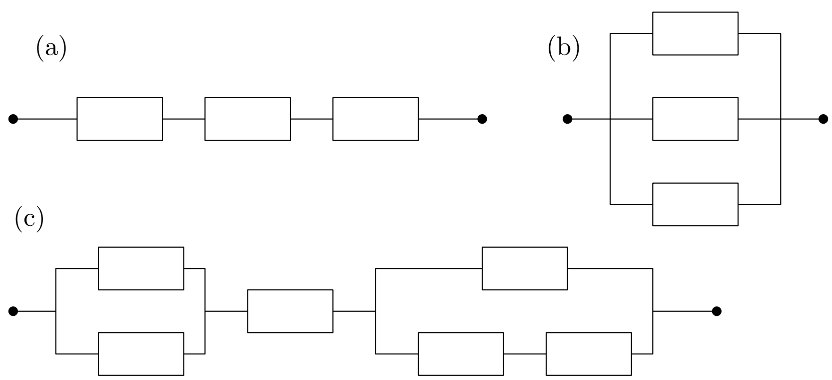
1st problem set for Probability and Statistics — February 18/19

Warm-up

1. Given a biased coin that comes up heads with some probability p, can we use it to simulate an unbiased coin toss? You can not pick the value of p, and you don't know the value, you only know that $0 < p < 1$. However, you are allowed to toss the coin multiple times.

2. Each rectangle in the figure is a component that can fail with probability p and then it doesn't conduct electricity. What is the probability that current still flows between the two dots?



3. In a class of 5 students, a nursery teacher has 5 pencil boxes, each labelled with a different student's name. The teacher randomly distributes one pencil box to each student. What is the probability that at least one student receives the pencil box with their own name?

Conditional probability

4. A student is taking a multiple choice exam in which each question has 5 possible answers, exactly one of which is correct. If the student knows the answer, she selects the correct answer. Otherwise, she chooses one option at random from the 5 possible answers. Suppose the student knows the answer to 70% of the questions.

(a) What is the probability that on a given question the student gets the correct answer?

(b) If the student gets the correct answer to a question, what is the probability that she knows the answer?

5. The king pardons two of the three prisoners Adam, Bob, and Cecil (and chooses at random). The warden offers to tell Adam a name of one pardoned prisoner (other than Adam). That is, if Adam and Bob are pardoned, he would tell Adam that Bob was pardoned, and similarly if Adam and Cecil are pardoned. If Bob and Cecil are pardoned, then the warden would pick one at random and tell Adam his name.

Adam refuses this offer though: he argues that if he knows the name, his own probability of release would be only 1/2, whereas now the probability is 2/3. Is he right?

6. Toss two coins: a dime and a nickel. Each of them comes up heads with probability p.

(a) What is the probability that both come up heads if the dime comes up heads?

(b) What is the probability that both come up heads if we know that at least one of them comes up heads?

Is it clear without counting which probability is greater?

Bonus problems

7. (Bertrand's paradox) For an equilateral triangle with side 1, circumscribe a circle around it, and pick a random chord of this circle. What is the probability that the length of this chord is greater than 1?

8. There are two envelopes on a table and we know that each envelope contains some (non-zero, integer) number of thousand crown notes, and the amount of money in the two envelopes is different. We are allowed to open one envelope and then decide whether to keep that one, or to switch to the other. If we want to get the envelope with the higher amount, can we achieve this with probability greater than 1/2? (See the second page for help.)

9. We have k containers, each containing a white and b black balls. We pick a random ball from the first, throw it into the second. Then we pick a random ball from it, throw it into the third, etc. What is the probability of getting a white ball out of the last container?

10. There are a black and b white balls in the urn. We draw balls from it one by one (without returning them). What is the probability that the first ball drawn is black? Second, third, ...?

More practice problems

11. Roll the dice twice. Let

- *Sum* be the event that the sum of the numbers thrown is 10,
- F be the event that the first roll of the dice is a six, and
- S be the event that a six is rolled on some roll (maybe both).

Calculate the probabilities of the given phenomena and all conditional probabilities.

12. We flip a coin and if it lands tails, we win the coin. We get a new coin for a new toss. We repeat the process until we get heads for the first time, then the game stops. We get all the coins that landed tails. (What is the probability that we get k coins?) After this we gather all the coins that we collected and toss them again, all at once. If all of them land tails, we can keep all of them. What is the probability of this happening?

13. Let's call Z to be the event that Quido will set up an automatic backup of important files over the next twelve months and D to be the event that Quido will lose an important file in the next 12 months.

(a) Which one would you guess to be bigger, $P(Z \mid D)$ or $P(Z \mid D^c)$?
(b) Do you think $P(D \mid Z)$ is bigger than $P(D \mid Z^c)$ or not?
(c) Use the definition to check whether your guess is mathematically possible.
(d) Think about why.

14. There are four dead batteries in a box of 100.

(a) What is the probability that if we pick three of the batteries at random for the headlamp, it will work? (They all have to be in in order.)

(b) What's the probability that at least two out of three are OK?

(c) We pull out a random battery three times, measure it, and (since it's OK) throw it back in. What's the probability of that happening?

(Without working out the numbers: is probability (a) higher or (c)?)

Help for the envelope problem: Pull out a coin and flip until it lands heads. Let X be the total number of flips (including the last one, i.e. $X \geq 1$.) If our envelope contains k notes, we keep the envelope if $X < k$.

What is the probability of getting an envelope with a higher amount?
