Probability and Statistics 1. Exercises 8

Convention: $[n]$ stands for $\{1, \ldots, n\}$. * indicates a bonus question for students interested to explore the topic in more depth.

1. A biased coin, which lands heads with probability $1/10$ each time it is flipped, is flipped 200 times consecutively. Give an upper bound on the probability that it lands heads at least 120 times using
   - Markov's inequality
   - Chebyshev's inequality
2. Let X and Y be two random variables with $E[X] = 1$, $\mathrm{Var}(X) = 4$, and $E[Y] = 2$, $\mathrm{Var}(Y) = 1$. Find the maximum possible value for $E[XY]$. Also, express Y as a function of X for which this maximum is achieved. (Hint: The best bound is not obtained by the Cauchy-Schwarz inequality, but by the application of correlation coefficient.)
3. Prove the central limit theorem for random variables, if $X_i \sim N(\mu, \sigma_i)$, where $i \in [n]$. [Hint: Use the convolution of normally distributed random variables.]
4. You're throwing a party for 100 guests and wondering how many sandwiches to order. You know from experience that the number of sandwiches eaten by a random guest follows a Poisson distribution with a mean of 3. Approximately how many sandwiches do you need to order so that with probability 0.95 no guest will go hungry? (Hint: Use an appropriate limit theorem.)
5. * Suppose that each of $m \geq 1$ pigeons independently and at random enter one of $n \geq 1$ pigeonholes. If $m \geq 1.2\sqrt{n} + 1$, then show that the probability that two pigeons go into the same pigeonhole is greater than $1/2$. (In particular, this implies that in a room having 25 people, the probability that at least two people have the same birthday is more than $1/2$.)
6. * If a sequence of random variables X_n converges to the random variable X almost surely, then prove that the sequence X_n converges to X in probability.