Probability and Statistics 1. Exercises 7

1. Let $X \sim N(0,1)$ and $Y = |X|$.
   1. What is the cumulative distribution function of Y?
   2. What is the probability density function of Y?
   3. Compute $E[Y]$.
2. Compute the convolution of continuous random variables X, Y if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$.
3. Let X and Y be jointly continuous random variables with finite mean and variance. Prove that $\rho(X,Y) = 1$ if and only if $Y = aX + b$ almost surely, for some $a > 0$. Prove that $\rho(X,Y) = -1$ if and only if $Y = aX + b$ almost surely, for some $a < 0$.
4. * The random graph $G(n,p)$ is an undirected (labeled) graph on n vertices such that each of the $\binom{n}{2}$ edges is present in the graph independently with probability p. That is, $G(n,p)$ defines a distribution over the set of undirected graphs on n vertices. If $G \sim G(n,p)$, i.e., G is a random graph with the $G(n,p)$ distribution then for every fixed graph G_0 on n vertices with m edges, we have $P[G = G_0] = p^m(1-p)^{\binom{n}{2}-m}$.

   For $G \sim G(n,p)$, show that

   (a) For p such that $\lim_{n\to\infty} pn = 0$, show that $\lim_{n\to\infty} P[G \text{ contains a triangle}] = 0$. (Here, we say that G does not contain a triangle with high probability.)

   (b) If $\lim_{n\to\infty} \frac{1}{pn} = 0$, show that $\lim_{n\to\infty} P[G \text{ contains a triangle}] = 1$. (Here, we say that G contains a triangle with high probability.)

   (Hint: Use Markov's and Chebyshev's inequalities.)