Probability and Statistics 1. Exercises 5

Convention: $[n]$ stands for $\{1, \ldots, n\}$. * indicates a bonus question for students interested to explore the topic in more depth.

1. We say that the random variables X and Y are uncorrelated if the covariance $C(X, Y) = 0$, where the covariance is defined as $C(X, Y) := E[XY] - E[X]E[Y]$. Define the random variable $X \in \{1, 2, 3\}$ with $P(X = i) = 1/3$ for $i = 1, 2, 3$. Consider the random variable $Y = 1$, if $X = 1$ and $Y = 0$, otherwise. Show that X and Y are uncorrelated. Are X and Y independent random variables?

2. Let X, Y be two independent random variables having Poisson distribution with parameters λ, μ respectively.

   - Find the distribution of the convolution $Z = X + Y$.
   - Calculate $E[Z(Z-1)]$. (Hint: Use LOTUS.)
   - Compute $\mathrm{Var}(Z)$.
   - If $\mu = 2\lambda$, find $C(X, Z)$.

3. Let g be a convex function and X be a random variable. Prove that $E[g(X)] \geq g(E[X])$. This is known as Jensen's inequality. [Hint: Assume $g(X)$ is a convex function and the linear function $f(X) = a + bX$ is tangential to $g(X)$ at the point $E[X]$. Recall that the graph of a convex function lies entirely above its tangent at every point. Now, use the linearity of expectation.]

4. * Consider a permutation of $[n]$ chosen uniformly at random from all possible permutations (as before). Let the random variable X be the number of fixed points in this random permutation. Find $E[X]$ and $\mathrm{Var}(X)$.