# Probability and Statistics 1. Exercises 3

Convention: $[n]$ stands for $\{1, \ldots, n\}$. * indicates a bonus question for students interested to explore the topic in more depth.

1. Three friends decide to go swimming on some day of a given week but don't arrange the day. So, each one shows up at the swimming pool on a (uniformly) random day, independently. Consider the random variable $X$ to be the number of people from these three who went on Friday. Find the probability distribution of $X$. Generalize this to $n$ friends.

2. Let $X \sim Geo(1/2)$. Show that $(-2)^X$ does not have an expectation.

3. Prove the following properties of the cumulative distribution function $F_X$ of a discrete random variable $X$

   (a) $F_X$ is non-decreasing

   (b) $\lim_{t \to -\infty} F_X(t) = 0$ and $\lim_{t \to \infty} F_X(t) = 1$

   (c) $F_X$ is right-continuous.

4. Let $X$ and $Y$ be discrete random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that

   (a) $f(X)$ is a discrete random variable,

   (b) $X + Y$ is a discrete random variable,

   (c) $\{X = Y\} \in \mathcal{F}$.

5. * Consider a permutation of $[n]$ chosen uniformly at random from all possible permutations. What is the probability that the permutation has exactly $k$ fixed points? Determine its limit when $k$ is fixed and $n \to \infty$.