Probability and Statistics 1. Exercises 2

Convention: $[n]$ stands for $\{1, \ldots, n\}$. * indicates a bonus question for students interested to explore the topic in more depth.

1. We pick 3 cards from the top of a randomly shuffled standard deck with 52 cards. What is the probability that none of the three cards is a spade?

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using the continuity of probability from the lecture prove the following:

   (a) Let $A_1 \supseteq A_2 \supseteq \ldots$ be a (decreasing) sequence of events. Show that

   $$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

   (b) Let $C_1, C_2, \ldots$ be a sequence of arbitrary events. Show that

   $$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} C_i\right), \quad \text{and} \quad \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} C_n\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{n} C_i\right).$$

3. Assume a Covid test is 95 percent effective in identifying the disease when a person is infected, whereas for non-infected people the test yields a negative result in 99 percent of the cases (and a false positive for the remaining 1 percent). If 0.5 percent of the population of a city actually have Covid, what is the probability a citizen is infected if their test was positive?

4. Let $\Omega = \{0, 1\}^n$, equipped with the uniform measure; this describes an n-fold toss of a fair coin. For each $i \in [n]$, let A_i be the event that the i-th throw was a 1. Let A_{n+1} be the event that the total number of 1's is even. Show that $\{A_1, \ldots, A_{n+1}\}$ are pairwise independent but not (mutually) independent.

5. * Analyse the game of gambler's ruin with a biased coin. That is, with a fixed probability $0 < p < 1$ the coin falls *heads* and A gets 1 CZK from B, with probability $1 - p$ the reverse happens. The rest of the setup is as in the unbiased version. What are the probabilities that player A/ player B wins the game?