Probability and Statistics 1. Exercises

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using the axioms of probability show that

   (a) $\mathbb{P}(A) + \mathbb{P}(\Omega \setminus A) = 1$ for all $A \in \mathcal{F}$.

   (b) $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$ for all $A, B \in \mathcal{F}$.

   (c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

   (d) $\mathbb{P}(\bigcup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \mathbb{P}(A_i)$ for any sequence $(A_i)_{i \in \mathbb{N}}$ of events.

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Fix an event $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$ and define a function $Q : \mathcal{F} \to [0, 1]$ by

$$Q(A) := \mathbb{P}(A \mid B).$$

   Prove that $(\Omega, \mathcal{F}, Q)$ is a probability space.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(A_i)_{i \in \mathbb{N}}$ be a sequence of events (i.e. members of $\mathcal{F}$). Prove that the following are events

   (a) Exactly one A_i holds. [1]

   (b) Infinitely many A_i hold.

   (c) All but finitely many A_i hold.

4. Let $S = \{0, 1\}$ and $\Omega = 2^{\mathbb{N}}$ – this describes an infinite sequence of coin tosses. Let $\mathcal{F}$ be a σ-algebra that contains all sets of the form $A_1 \times \cdots \times A_k \times S \times S \times \ldots$, where $k \in \mathbb{N}$ and $A_1, \ldots A_k$ are subsets of S.

   (a) Let $Fair$ be the set of all of infinite binary strings in which the proportion of 1's converges to 1/2 (under processing the string 'left to right'). Prove that $Fair \in \mathcal{F}$.

   (b) Similarly, let $Conv$ be the set of all strings in which the proportion of 1's converges (to an unspecified value in $[0, 1]$). Prove that $Conv \in \mathcal{F}$. **Hint:** Analysis 1.

[1] This stands for $\{\omega \in \Omega : \omega$ belongs to exactly one of the $A_i\}$, and similarly for the other questions.