Probability and Statistics 1. Practice problems

1. We have five keys on a ring, one of them is correct, but we don't know which one. We try to open the door.

   - After each attempt, the ring slips away, and we always choose again at random.
   - We select in random order, but each key only once (we can mark them).

   In both the cases, let the random variable be the number of attempts to open the door. Name the distribution of this random variable and calculate its mean.

   - As part (a), but two keys out of ten are correct?
   - As part (b), but two keys out of ten are correct? (Here, determining the mean value is a little more difficult, so just specify the probability function.)

2. You are in a group with 500 other people. Let X be the number of people with the same birthday. (Ignore leap years and the fact that not all days have the same number of children sharing their birthday.)

   - Find $p_X(1)$.
   - What is the distribution of X?
   - What is the mean value?
   - Approximate $p_X(1)$ using the Poisson distribution.
   - If Y is a similar count for your neighbour – what is the relationship between X and Y? [Hint: you can start with $p_X(0)$, that's a bit easier.]

3. X is a uniformly random power of two from the set $\{2^a, 2^{a+1}, \ldots, 2^b\}$.

   - Express X using a random variable U that is uniformly distributed over the set $\{a, a+1, \ldots, b\}$.
   - Determine $E[X]$ and $\mathrm{Var}(X)$.

4. Let X_1, X_2, X_3 denote the results of three independent rolls with a four-sided die (i.e., the outcomes are the numbers 1, 2, 3, or 4).

   - What is the probability distribution of $X = X_1$?
   - What is the probability distribution of $Y = \max(X_1, X_2)$?
   - What is the probability distribution of $Z = \max(X_1, X_2, X_3)$?

5. We toss a coin three times. We denote by X the number of heads in the first two tosses and Y the number of tails in the last two tosses.

   - Determine the joint probability function $p_{X,Y}$ as well as the marginal probability function p_X, p_Y.
   - Are X and Y independent?
   - Determine $P(X < Y)$.
   - Determine the conditional probability function $p_{X|Y}$, i.e. the numbers $P(X = x|Y = y)$ for all values of x, y.