Homework 1: Probability and Statistics 1

Due: 3 April 2023

Each problem is worth 3 points for a total of 15 points. Solutions must be submitted by 3rd April 2023 23:59 in ONE pdf file (NO .jpeg files) by email (to gaurav@kam.mff.cuni.cz). LaTeX write-ups are preferred, but handwritten solutions will be accepted as long as they are legible and scanned. You are allowed to discuss the problems with other students, but every student must submit their own solutions!

1. Let us have an undirected fixed graph $G = (V, E)$. We randomly select a subset of vertices $V' \subseteq V$ by tossing a fair coin for each vertex, whether to put it in V'. What is the expectation of the number of edges that run between V' and $V \setminus V'$?

2. We roll three dice and read the result as a three-digit number, i.e., if the first dice rolls a 4, the second rolls a 6 and the last rolls a 2 then the result is 462. What is the expected value of the results?

3. Consider the sample space obtained upon rolling two dice. Give examples of events A, B where (a) $P[A \mid B] < P[A]$, (b) $P[A \mid B] = P[A]$, and (c) $P[A \mid B] > P[A]$.

4. A random variable X follows the *hypergeometric distribution* if its probability mass function (pmf) is given by

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.$$

   Compute $E[X]$, $\mathrm{Var}[X]$.

5. Let X and Y be random variables. We define the conditional expectation of X given Y as

$$E[X \mid Y = y] := \sum_x xP(X = x \mid Y = y).$$

   Consider any finite collection of discrete random variables $X_1, X_2, ..., X_n$ with finite expectations and a random variable Y. Show that

$$E\left[\sum_{i=1}^{n} X_i \mid Y = y\right] = \sum_{i=1}^{n} E[X_i \mid Y = y]$$